(MODIFIED) FREDHOLM DETERMINANTS FOR OPERATORS WITH MATRIX-VALUED SEMI-SEPARABLE INTEGRAL KERNELS REVISITED

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Dedicated with great pleasure to Eduard R. Tsekanovskii on the occasion of his 65th birthday.

Abstract. We revisit the computation of (2-modified) Fredholm determinants for operators with matrix-valued semi-separable integral kernels. The latter occur, for instance, in the form of Green's functions associated with closed ordinary differential operators on arbitrary intervals on the real line. Our approach determines the (2-modified) Fredholm determinants in terms of solutions of closely associated Volterra integral equations, and as a result offers a natural way to compute such determinants.

We illustrate our approach by identifying classical objects such as the Jost function for half-line Schrödinger operators and the inverse transmission coefficient for Schrödinger operators on the real line as Fredholm determinants, and rederiving the well-known expressions for them in due course. We also apply our formalism to Floquet theory of Schrödinger operators, and upon identifying the connection between the Floquet discriminant and underlying Fredholm determinants, we derive new representations of the Floquet discriminant.

Finally, we rederive the explicit formula for the 2-modified Fredholm determinant corresponding to a convolution integral operator, whose kernel is associated with a symbol given by a rational function, in a straghtforward manner. This determinant formula represents a Wiener–Hopf analog of Day's formula for the determinant associated with finite Toeplitz matrices generated by the Laurent expansion of a rational function.

1. Introduction

We offer a self-contained and elementary approach to the computation of Fredholm and 2-modified Fredholm determinants associated with $m \times m$ matrix-valued, semi-separable integral kernels on arbitrary intervals $(a,b) \subseteq \mathbb{R}$ of the type

$$K(x,x') = \begin{cases} f_1(x)g_1(x'), & a < x' < x < b, \\ f_2(x)g_2(x'), & a < x < x' < b, \end{cases} \tag{1.1}$$

associated with the Hilbert–Schmidt operator K in $L^2((a,b);dx)^m$, $m \in \mathbb{N}$,

$$(Kf)(x) = \int_a^b dx' \, K(x,x')f(x'), \quad f \in L^2((a,b);dx)^m, \tag{1.2}$$

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assuming

$$f_j \in L^2((a,b);dx)^{m\times n_j},\ g_j \in L^2((a,b);dx)^{n_j\times m}, \quad n_j\in\mathbb{N},\ j=1,2. \tag{1.3}$$

We emphasize that Green's matrices and resolvent operators associated with closed ordinary differential operators on arbitrary intervals (finite or infinite) on the real line are always of the form (1.1)–(1.3) (cf. [11, Sect. XIV.3]), as are certain classes of convolution operators (cf. [11, Sect. XIII.10]).

To describe the approach of this paper we briefly recall the principal ideas of the approach to $m\times m$ matrix-valued semi-separable integral kernels in the monographs by Gohberg, Goldberg, and Kaashoek [11, Ch. IX] and Gohberg, Goldberg, and Krupnik [14, Ch. XIII]. It consists in decomposing K in (1.2) into a Volterra operator H_a and a finite-rank operator QR

$$K = H_a + QR, \tag{1.4}$$

where

$$(H_a f)(x) = \int_a^x dx'\, H(x,x')f(x'), \quad f\in L^2((a,b);dx)^m, \tag{1.5}$$

$$H(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x'), \quad a<x'<x<b \tag{1.6}$$

and

$$Q\colon \mathbb{C}^{n_2}\mapsto L^2((a,b);dx)^m, \quad (Q\underline{u})(x) = f_2(x)\underline{u}, \quad \underline{u}\in\mathbb{C}^{n_2}, \tag{1.7}$$

$$R\colon L^2((a,b);dx)^m\mapsto \mathbb{C}^{n_2}, \quad (Rf) = \int_a^b dx'\, g_2(x')f(x'), \quad f\in L^2((a,b);dx)^m. \tag{1.8}$$

Moreover, introducing

$$C(x) = (f_1(x)\ \ f_2(x)), \quad B(x) = (g_1(x)\ \ -g_2(x))^\top \tag{1.9}$$

and the $n\times n$ matrix A ($n=n_1+n_2$)

$$A(x) = \begin{pmatrix} g_1(x)f_1(x) & g_1(x)f_2(x) \\ -g_2(x)f_1(x) & -g_2(x)f_2(x)\end{pmatrix}, \tag{1.10}$$

one considers a particular nonsingular solution $U(\cdot,\alpha)$ of the following first-order system of differential equations

$$U'(x,\alpha) = \alpha A(x)U(x,\alpha) \text{ for a.e. } x\in(a,b) \text{ and } \alpha\in\mathbb{C} \tag{1.11}$$

and obtains

$$(I-\alpha H_a)^{-1} = I + \alpha J_a(\alpha) \text{ for all } \alpha\in\mathbb{C}, \tag{1.12}$$

$$(J_a(\alpha)f)(x) = \int_a^x dx'\, J(x,x',\alpha)f(x'), \quad f\in L^2((a,b);dx)^m, \tag{1.13}$$

$$J(x,x',\alpha) = C(x)U(x,\alpha)U(x',\alpha)^{-1}B(x'), \quad a<x'<x<b. \tag{1.14}$$

Next, observing

$$I-\alpha K = (I-\alpha H_a)[I-\alpha(I-\alpha H_a)^{-1}QR] \tag{1.15}$$

and assuming that K is a trace class operator,

$$K\in\mathcal{B}_1(L^2((a,b);dx)^m), \tag{1.16}$$

one computes,

$$\begin{aligned}\det(I-\alpha K) &= \det(I-\alpha H_a)\det(I-\alpha(I-\alpha H_a)^{-1}QR) \\ &= \det(I-\alpha(I-\alpha H_a)^{-1}QR) \\ &= \det{}_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q). \end{aligned} \tag{1.17}$$

In particular, the Fredholm determinant of $I-\alpha K$ is reduced to a finite-dimensional determinant induced by the finite rank operator QR in (1.4). Up to this point we followed the treatment in [11, Ch. IX]). Now we will depart from the presentation in [11, Ch. IX] and [14, Ch. XIII] that focuses on a solution $U(\cdot,\alpha)$ of (1.11) normalized by $U(a,\alpha)=I_n$. The latter normalization is in general not satisfied for Schrödinger operators on a half-line or on the whole real line possessing eigenvalues as discussed in Section 4.

To describe our contribution to this circle of ideas we now introduce the Volterra integral equations

$$\begin{aligned}\hat f_1(x,\alpha) &= f_1(x)-\alpha\int_x^b dx'\,H(x,x')\hat f_1(x',\alpha), \\ \hat f_2(x,\alpha) &= f_2(x)+\alpha\int_a^x dx'\,H(x,x')\hat f_2(x',\alpha), \quad \alpha\in\mathbb{C}\end{aligned} \tag{1.18}$$

with solutions $\hat f_j(\cdot,\alpha)\in L^2((a,b);dx)^{m\times n_j}$, $j=1,2$, and note that the first-order $n\times n$ system of differential equations (1.11) then permits the explicit particular solution

$$U(x,\alpha)=\begin{pmatrix} I_{n_1}-\alpha\int_x^b dx'\,g_1(x')\hat f_1(x',\alpha) & \alpha\int_a^x dx'\,g_1(x')\hat f_2(x',\alpha) \\ \alpha\int_x^b dx'\,g_2(x')\hat f_1(x',\alpha) & I_{n_2}-\alpha\int_a^x dx'\,g_2(x')\hat f_2(x',\alpha)\end{pmatrix}, \quad x\in(a,b). \tag{1.19}$$

Given (1.19), one can supplement (1.17) by

$$\begin{aligned}\det(I-\alpha K) &= \det{}_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q) \\ &= \det{}_{\mathbb{C}^{n_2}}\left(I_{n_2}-\alpha\int_a^b dx\,g_2(x)\hat f_2(x,\alpha)\right) \\ &= \det{}_{\mathbb{C}^{n}}(U(b,\alpha)),\end{aligned} \tag{1.20}$$

our principal result. A similar set of results can of course be obtained by introducing the corresponding Volterra operator H_b in (2.5). Moreover, analogous results hold for 2-modified Fredholm determinants in the case where K is only assumed to be a Hilbert–Schmidt operator.

Equations (1.17) and (1.20) summarize this approach based on decomposing K into a Volterra operator plus finite rank operator in (1.4), as advocated in [11, Ch. IX] and [14, Ch. XIII], and our additional twist of relating this formalism to the underlying Volterra integral equations (1.18) and the explicit solution (1.19) of (1.11).

In Section 2 we set up the basic formalism leading up to the solution U in (1.19) of the first-order system of differential equations (1.11). In Section 3 we derive the set of formulas (1.17), (1.20), if K is a trace class operator, and their counterparts for 2-modified Fredholm determinants, assuming K to be a Hilbert–Schmidt operator only. Section 4 then treats four particular applications: First we treat the case of half-line Schrödinger operators in which we identify the Jost

function as a Fredholm determinant (a well-known, in fact, classical result due to Jost and Pais [23]). Next, we study the case of Schrödinger operators on the real line in which we characterize the inverse of the transmission coefficient as a Fredholm determinant (also a well-known result, see, e.g., [31, Appendix A], [36, Proposition 5.7]). We also revisit this problem by replacing the second-order Schrödinger equation by the equivalent first-order 2×2 system and determine the associated 2-modified Fredholm determinant. The case of periodic Schrödinger operators in which we derive a new one-parameter family of representations of the Floquet discriminant and relate it to underlying Fredholm determinants is discussed next. Apparently, this is a new result. In our final Section 5, we rederive the explicit formula for the 2-modified Fredholm determinant corresponding to a convolution integral operator whose kernel is associated with a symbol given by a rational function. The latter represents a Wiener–Hopf analog of Day's formula [7] for the determinant of finite Toeplitz matrices generated by the Laurent expansion of a rational function. The approach to (2-modified) Fredholm determinants of semi-separable kernels advocated in this paper permits a remarkably elementary derivation of this formula compared to the current ones in the literature (cf. the references provided at the end of Section 5).

The effectiveness of the approach pursued in this paper is demonstrated by the ease of the computations involved and by the unifying character it takes on when applied to differential and convolution-type operators in several different settings.

2. Hilbert–Schmidt operators with semi-separable integral kernels

In this section we consider Hilbert-Schmidt operators with matrix-valued semi-separable integral kernels following Gohberg, Goldberg, and Kaashoek [11, Ch. IX] and Gohberg, Goldberg, and Krupnik [14, Ch. XIII] (see also [15]). To set up the basic formalism we introduce the following hypothesis assumed throughout this section.

Hypothesis 2.1. *Let $-\infty \le a < b \le \infty$ and $m, n_1, n_2 \in \mathbb{N}$. Suppose that f_j are $m \times n_j$ matrices and g_j are $n_j \times m$ matrices, $j = 1, 2$, with* (*Lebesgue*) *measurable entries on* (a, b) *such that*

$$f_j \in L^2((a,b);dx)^{m\times n_j},\ g_j \in L^2((a,b);dx)^{n_j\times m}, \quad j=1,2. \tag{2.1}$$

Given Hypothesis 2.1, we introduce the Hilbert–Schmidt operator

$$\begin{aligned} &K \in \mathcal{B}_2(L^2((a,b);dx)^m), \\ &(Kf)(x) = \int_a^b dx'\, K(x,x')f(x'), \quad f \in L^2((a,b);dx)^m \end{aligned} \tag{2.2}$$

in $L^2((a,b);dx)^m$ with $m \times m$ matrix-valued integral kernel $K(\cdot,\cdot)$ defined by

$$K(x,x') = \begin{cases} f_1(x)g_1(x'), & a < x' < x < b, \\ f_2(x)g_2(x'), & a < x < x' < b. \end{cases} \tag{2.3}$$

One verifies that K is a finite rank operator in $L^2((a,b);dx)^m$ if $f_1 = f_2$ and $g_1 = g_2$ a.e. Conversely, any finite rank operator in $L^2((a,b));dx)^m$ is of the form (2.2), (2.3) with $f_1 = f_2$ and $g_1 = g_2$ (cf. [11, p. 150]).

Associated with K we also introduce the Volterra operators H_a and H_b in $L^2((a,b);dx)^m$ defined by

$$(H_a f)(x) = \int_a^x dx'\, H(x,x')f(x'), \tag{2.4}$$

$$(H_b f)(x) = -\int_x^b dx'\, H(x,x')f(x'); \quad f \in L^2((a,b);dx)^m, \tag{2.5}$$

with $m \times m$ matrix-valued (triangular) integral kernel

$$H(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x'). \tag{2.6}$$

Moreover, introducing the matrices[1]

$$C(x) = (f_1(x) \;\; f_2(x)), \tag{2.7}$$

$$B(x) = (g_1(x) \;\; -g_2(x))^\top, \tag{2.8}$$

one verifies

$$H(x,x') = C(x)B(x'), \;\text{ where } \begin{cases} a < x' < x < b & \text{for } H_a, \\ a < x < x' < b & \text{for } H_b \end{cases} \tag{2.9}$$

and[2]

$$K(x,x') = \begin{cases} C(x)(I_n - P_0)B(x'), & a < x' < x < b, \\ -C(x)P_0B(x'), & a < x < x' < b \end{cases} \tag{2.10}$$

with

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}. \tag{2.11}$$

Next, introducing the linear maps

$$Q\colon \mathbb{C}^{n_2} \mapsto L^2((a,b);dx)^m, \quad (Q\underline{u})(x) = f_2(x)\underline{u}, \quad \underline{u} \in \mathbb{C}^{n_2}, \tag{2.12}$$

$$R\colon L^2((a,b);dx)^m \mapsto \mathbb{C}^{n_2}, \quad (Rf) = \int_a^b dx'\, g_2(x')f(x'), \quad f \in L^2((a,b);dx)^m, \tag{2.13}$$

$$S\colon \mathbb{C}^{n_1} \mapsto L^2((a,b);dx)^m, \quad (S\underline{v})(x) = f_1(x)\underline{v}, \quad \underline{v} \in \mathbb{C}^{n_1}, \tag{2.14}$$

$$T\colon L^2((a,b);dx)^m \mapsto \mathbb{C}^{n_1}, \quad (Tf) = \int_a^b dx'\, g_1(x')f(x'), \quad f \in L^2((a,b);dx)^m, \tag{2.15}$$

one easily verifies the following elementary yet significant result.

Lemma 2.2 ([11], Sect. IX.2; [14], Sect. XIII.6)**.** *Assume Hypothesis 2.1. Then*

$$K = H_a + QR \tag{2.16}$$

$$= H_b + ST. \tag{2.17}$$

In particular, since R and T are of finite rank, so are $K - H_a$ and $K - H_b$.

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[1] $M^\top$ denotes the transpose of the matrix M.

[2] I_k denotes the identity matrix in $\mathbb{C}^k$, $k \in \mathbb{N}$.

Remark 2.3. *The decompositions* (2.16) *and* (2.17) *of K are significant since they prove that K is the sum of a Volterra and a finite rank operator. As a consequence, the* (2*-modified*) *determinants corresponding to $I-\alpha K$ can be reduced to determinants of finite-dimensional matrices, as will be further discussed in Sections 3 and 4.*

To describe the inverse[3] of $I-\alpha H_a$ and $I-\alpha H_b$, $\alpha\in\mathbb{C}$, one introduces the $n\times n$ matrix A $(n=n_1+n_2)$

$$A(x)=\begin{pmatrix} g_1(x)f_1(x) & g_1(x)f_2(x) \\ -g_2(x)f_1(x) & -g_2(x)f_2(x)\end{pmatrix} \tag{2.18}$$

$$= B(x)C(x) \text{ for a.e. } x\in(a,b) \tag{2.19}$$

and considers a particular nonsingular solution $U=U(x,\alpha)$ of the first-order $n\times n$ system of differential equations

$$U'(x,\alpha)=\alpha A(x)U(x,\alpha) \text{ for a.e. } x\in(a,b) \text{ and } \alpha\in\mathbb{C}. \tag{2.20}$$

Since $A\in L^1((a,b))^{n\times n}$, the general solution V of (2.20) is an $n\times n$ matrix with locally absolutely continuous entries on (a,b) of the form $V=UD$ for any constant $n\times n$ matrix D (cf. [11, Lemma IX.2.1])[4].

Theorem 2.4 ([11], Sect. IX.2; [14], Sects. XIII.5, XIII.6)**.**
Assume Hypothesis 2.1 and let $U(\cdot,\alpha)$ denote a nonsingular solution of (2.20)*. Then,*
(i) $I-\alpha H_a$ and $I-\alpha H_b$ are invertible for all $\alpha\in\mathbb{C}$ and

$$(I-\alpha H_a)^{-1}=I+\alpha J_a(\alpha), \tag{2.21}$$

$$(I-\alpha H_b)^{-1}=I+\alpha J_b(\alpha), \tag{2.22}$$

$$(J_a(\alpha)f)(x)=\int_a^x dx'\, J(x,x',\alpha)f(x'), \tag{2.23}$$

$$(J_b(\alpha)f)(x)=-\int_x^b dx'\, J(x,x',\alpha)f(x'); \quad f\in L^2((a,b);dx)^m, \tag{2.24}$$

$$J(x,x',\alpha)=C(x)U(x,\alpha)U(x',\alpha)^{-1}B(x'), \ \ \textit{where } \begin{cases} a<x'<x<b & \textit{for } J_a, \\ a<x<x'<b & \textit{for } J_b. \end{cases} \tag{2.25}$$

(ii) Let $\alpha\in\mathbb{C}$. Then $I-\alpha K$ is invertible if and only if the $n_2\times n_2$ matrix $I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q$ is. Similarly, $I-\alpha K$ is invertible if and only if the $n_1\times n_1$

[3] I denotes the identity operator in $L^2((a,b);dx)^m$.

[4] If $a>-\infty$, V extends to an absolutely continuous $n\times n$ matrix on all intervals of the type $[a,c)$, $c<b$. The analogous consideration applies to the endpoint b if $b<\infty$.

matrix $I_{n_1} - \alpha T(I-\alpha H_b)^{-1}S$ *is. In particular,*

$$\begin{aligned}(I-\alpha K)^{-1} &= (I-\alpha H_a)^{-1} + \alpha(I-\alpha H_a)^{-1}QR(I-\alpha K)^{-1} &(2.26)\\ &= (I-\alpha H_a)^{-1}\\ &\quad + \alpha(I-\alpha H_a)^{-1}Q[I_{n_2} - \alpha R(I-\alpha H_a)^{-1}Q]^{-1}R(I-\alpha H_a)^{-1} &(2.27)\\ &= (I-\alpha H_b)^{-1} + \alpha(I-\alpha H_b)^{-1}ST(I-\alpha K)^{-1} &(2.28)\\ &= (I-\alpha H_b)^{-1}\\ &\quad + \alpha(I-\alpha H_b)^{-1}S[I_{n_1} - \alpha T(I-\alpha H_b)^{-1}S]^{-1}T(I-\alpha H_b)^{-1}. &(2.29)\end{aligned}$$

Moreover,

$$(I-\alpha K)^{-1} = I + \alpha L(\alpha), \tag{2.30}$$

$$(L(\alpha)f)(x) = \int_a^b dx'\, L(x,x',\alpha)f(x'), \tag{2.31}$$

$$L(x,x',\alpha) = \begin{cases} C(x)U(x,\alpha)(I-P(\alpha))U(x',\alpha)^{-1}B(x'), & a < x' < x < b,\\ -C(x)U(x,\alpha)P(\alpha)U(x',\alpha)^{-1}B(x'), & a < x < x' < b,\end{cases} \tag{2.32}$$

where $P(\alpha)$ *satisfies*

$$P_0U(b,\alpha)(I-P(\alpha)) = (I-P_0)U(a,\alpha)P(\alpha), \quad P_0 = \begin{pmatrix} 0 & 0\\ 0 & I_{n_2}\end{pmatrix}. \tag{2.33}$$

Remark 2.5. (i) *The results* (2.21)–(2.25) *and* (2.30)–(2.33) *are easily verified by computing* $(I-\alpha H_a)(I+\alpha J_a)$ *and* $(I+\alpha J_a)(I-\alpha H_a)$, *etc., using an integration by parts. Relations* (2.26)–(2.29) *are clear from* (2.16) *and* (2.17), *a standard resolvent identity, and the fact that* $K-H_a$ *and* $K-H_b$ *factor into* QR *and* ST, *respectively.*
(ii) *The discussion in* [11, Sect. IX.2], [14, Sects. XIII.5, XIII.6] *starts from the particular normalization*

$$U(a,\alpha) = I_n \tag{2.34}$$

of a solution U *satisfying* (2.20). *In this case the explicit solution for* $P(\alpha)$ *in* (2.33) *is given by*

$$P(\alpha) = \begin{pmatrix} 0 & 0\\ U_{2,2}(b,\alpha)^{-1}U_{2,1}(b,\alpha) & I_{n_2}\end{pmatrix}. \tag{2.35}$$

However, for concrete applications to differential operators to be discussed in Section 4, the normalization (2.34) *is not necessarily possible.*

Rather than solving the basic first-order system of differential equations $U' = \alpha AU$ in (2.20) with the fixed initial condition $U(a,\alpha) = I_n$ in (2.34), we now derive an explicit particular solution of (2.20) in terms of closely associated solutions of Volterra integral equations involving the integral kernel $H(\cdot,\cdot)$ in (2.6). This approach is most naturally suited for the applications to Jost functions, transmission coefficients, and Floquet discriminants we discuss in Section 4 and to the class of Wiener–Hopf operators we study in Section 5.

Still assuming Hypothesis 2.1, we now introduce the Volterra integral equations

$$\hat{f}_1(x,\alpha) = f_1(x) - \alpha \int_x^b dx'\, H(x,x')\hat{f}_1(x',\alpha), \tag{2.36}$$

$$\hat{f}_2(x,\alpha) = f_2(x) + \alpha \int_a^x dx'\, H(x,x')\hat{f}_2(x',\alpha); \quad \alpha \in \mathbb{C}, \tag{2.37}$$

with solutions $\hat{f}_j(\cdot,\alpha) \in L^2((a,b);dx)^{m\times n_j}$, $j = 1,2$.

Lemma 2.6. *Assume Hypothesis 2.1 and let* $\alpha \in \mathbb{C}$.
(i) The first-order $n\times n$ *system of differential equations* $U' = \alpha A U$ *a.e. on* (a,b) *in* (2.20) *permits the explicit particular solution*

$$U(x,\alpha) = \begin{pmatrix} I_{n_1} - \alpha \int_x^b dx'\, g_1(x')\hat{f}_1(x',\alpha) & \alpha \int_a^x dx'\, g_1(x')\hat{f}_2(x',\alpha) \\ \alpha \int_x^b dx'\, g_2(x')\hat{f}_1(x',\alpha) & I_{n_2} - \alpha \int_a^x dx'\, g_2(x')\hat{f}_2(x',\alpha) \end{pmatrix}, \quad x \in (a,b). \tag{2.38}$$

As long as[5]

$$\det_{\mathbb{C}^{n_1}}\left(I_{n_1} - \alpha \int_a^b dx\, g_1(x)\hat{f}_1(x,\alpha)\right) \neq 0, \tag{2.39}$$

or equivalently,

$$\det_{\mathbb{C}^{n_2}}\left(I_{n_2} - \alpha \int_a^b dx\, g_2(x)\hat{f}_2(x,\alpha)\right) \neq 0, \tag{2.40}$$

U *is nonsingular for all* $x \in (a,b)$ *and the general solution* V *of* (2.20) *is then of the form* $V = UD$ *for any constant* $n\times n$ *matrix* D.
(ii) Choosing (2.38) *as the particular solution* U *in* (2.30)–(2.33), $P(\alpha)$ *in* (2.33) *simplifies to*

$$P(\alpha) = P_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix}. \tag{2.41}$$

Proof. Differentiating the right-hand side of (2.38) with respect to x and using the Volterra integral equations (2.36), (2.37) readily proves that U satisfies $U' = \alpha A U$ a.e. on (a,b).

By Liouville's formula (cf., e.g., [21, Theorem IV.1.2]) one infers

$$\det_{\mathbb{C}^n}(U(x,\alpha)) = \det_{\mathbb{C}^n}(U(x_0,\alpha)) \exp\left(\alpha \int_{x_0}^x dx'\, \mathrm{tr}_{\mathbb{C}^n}(A(x'))\right), \quad x, x_0 \in (a,b). \tag{2.42}$$

Since $\mathrm{tr}_{\mathbb{C}^n}(A) \in L^1((a,b);dx)$ by (2.1),

$$\lim_{x\downarrow a} \det_{\mathbb{C}^n}(U(x,\alpha)) \text{ and } \lim_{x\uparrow b} \det_{\mathbb{C}^n}(U(x,\alpha)) \text{ exist.} \tag{2.43}$$

Hence, if (2.39) holds, $U(x,\alpha)$ is nonsingular for x in a neighborhood (a,c), $a<c$, of a, and similarly, if (2.40) holds, $U(x,\alpha)$ is nonsingular for x in a neighborhood (c,b), $c<b$, of b. In either case, (2.42) then proves that $U(x,\alpha)$ is nonsingular for all $x \in (a,b)$.

Finally, since $U_{2,1}(b,\alpha) = 0$, (2.41) follows from (2.35). □

[5] $\det_{\mathbb{C}^k}(M)$ and $\mathrm{tr}_{\mathbb{C}^k}(M)$ denote the determinant and trace of a $k\times k$ matrix M with complex-valued entries, respectively.

Remark 2.7. *In concrete applications (e.g., to Schrödinger operators on a half-line or on the whole real axis as discussed in Section 4), it may happen that* $\det_{\mathbb{C}^n}(U(x,\alpha))$ *vanishes for certain values of intrinsic parameters (such as the energy parameter). Hence, a normalization of the type* $U(a,\alpha)=I_n$ *is impossible in the case of such parameter values and the normalization of* U *is best left open as illustrated in Section 4. One also observes that in general our explicit particular solution* U *in* (2.38) *satisfies* $U(a,\alpha)\neq I_n$, $U(b,\alpha)\neq I_n$.

Remark 2.8. *In applications to Schrödinger and Dirac-type systems,* A *is typically of the form*

$$A(x)=e^{-Mx}\widetilde{A}(x)e^{Mx}, \quad x\in(a,b) \tag{2.44}$$

where M *is an* x*-independent* $n\times n$ *matrix (in general depending on a spectral parameter) and* $\widetilde{A}$ *has a simple asymptotic behavior such that for some* $x_0\in(a,b)$

$$\int_a^{x_0} w_a(x)dx\,|\widetilde{A}(x)-\widetilde{A}_-|+\int_{x_0}^b w_b(x)dx\,|\widetilde{A}(x)-\widetilde{A}_+|<\infty \tag{2.45}$$

for constant $n\times n$ *matrices* $\widetilde{A}_\pm$ *and appropriate weight functions* $w_a\geq 0$, $w_b\geq 0$. *Introducing* $W(x,\alpha)=e^{Mx}U(x,\alpha)$, *equation* (2.20) *reduces to*

$$W'(x,\alpha)=[M+\alpha\widetilde{A}(x)]W(x,\alpha), \quad x\in(a,b) \tag{2.46}$$

with

$$\det_{\mathbb{C}^n}(W(x,\alpha))=\det_{\mathbb{C}^n}(U(x,\alpha))e^{-\operatorname{tr}_{\mathbb{C}^n}(M)x}, \quad x\in(a,b). \tag{2.47}$$

The system (2.46) *then leads to operators* H_a, H_b, *and* K. *We will briefly illustrate this in connection with Schrödinger operators on the line in Remark 4.8.*

3. (Modified) Fredholm determinants for operators with semi-separable integral kernels

In the first part of this section we suppose that K is a trace class operator and consider the Fredholm determinant of $I-K$. In the second part we consider 2-modified Fredholm determinants in the case where K is a Hilbert–Schmidt operator.

In the context of trace class operators we assume the following hypothesis.

Hypothesis 3.1. *In addition to Hypothesis 2.1, we suppose that* K *is a trace class operator,* $K\in\mathcal{B}_1(L^2((a,b);dx)^m)$.

The following results can be found in Gohberg, Goldberg, and Kaashoek [11, Theorem 3.2] and in Gohberg, Goldberg, and Krupnik [14, Sects. XIII.5, XIII.6] under the additional assumptions that a,b are finite and U satisfies the normalization $U(a)=I_n$ (cf. (2.20), (2.34)). Here we present the general case where $(a,b)\subseteq\mathbb{R}$ is an arbitrary interval on the real line and U is not normalized but given by the particular solution (2.38).

In the course of the proof we use some of the standard properties of determinants, such as,

$$\det((I_{\mathcal{H}}-A)(I_{\mathcal{H}}-B))=\det(I_{\mathcal{H}}-A)\det(I_{\mathcal{H}}-B), \quad A,B\in\mathcal{B}_1(\mathcal{H}), \tag{3.1}$$

$$\det(I_{\mathcal{H}_1}-AB)=\det(I_{\mathcal{H}}-BA) \quad \text{for all } A\in\mathcal{B}_1(\mathcal{H}_1,\mathcal{H}),\ B\in\mathcal{B}(\mathcal{H},\mathcal{H}_1) \tag{3.2}$$
$$\text{such that } AB\in\mathcal{B}_1(\mathcal{H}_1),\ BA\in\mathcal{B}_1(\mathcal{H}),$$

and

$$\det(I_{\mathcal{H}} - A) = \det_{\mathbb{C}^k}(I_k - D_k) \ \text{ for } \ A = \begin{pmatrix} 0 & C \\ 0 & D_k \end{pmatrix}, \ \mathcal{H} = \mathcal{K} \dotplus \mathbb{C}^k, \tag{3.3}$$

since

$$I_{\mathcal{H}} - A = \begin{pmatrix} I_{\mathcal{K}} & -C \\ 0 & I_k - D_k \end{pmatrix} = \begin{pmatrix} I_{\mathcal{K}} & 0 \\ 0 & I_k - D_k \end{pmatrix} \begin{pmatrix} I_{\mathcal{K}} & -C \\ 0 & I_k \end{pmatrix}. \tag{3.4}$$

Here $\mathcal{H}$ and $\mathcal{H}_1$ are complex separable Hilbert spaces, $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$, $\mathcal{B}_p(\mathcal{H})$, $p \geq 1$, denote the usual trace ideals of $\mathcal{B}(\mathcal{H})$, and $I_{\mathcal{H}}$ denotes the identity operator in $\mathcal{H}$. Moreover, $\det_p(I_{\mathcal{H}} - A)$, $A \in \mathcal{B}_p(\mathcal{H})$, denotes the ($p$-modified) Fredholm determinant of $I_{\mathcal{H}} - A$ with $\det_1(I_{\mathcal{H}} - A) = \det(I_{\mathcal{H}} - A)$, $A \in \mathcal{B}_1(\mathcal{H})$, the standard Fredholm determinant of a trace class operator, and $\operatorname{tr}(A)$, $A \in \mathcal{B}_1(\mathcal{H})$, the trace of a trace class operator. Finally, $\dotplus$ in (3.3) denotes a direct but not necessary orthogonal direct decomposition of $\mathcal{H}$ into $\mathcal{K}$ and the k-dimensional subspace $\mathbb{C}^k$. (We refer, e.g., to [12], [18, Sect. IV.1], [34, Ch. 17], [35], [36, Ch. 3] for these facts).

Theorem 3.2. *Suppose Hypothesis 3.1 and let $\alpha \in \mathbb{C}$. Then,*

$$\operatorname{tr}(H_a) = \operatorname{tr}(H_b) = 0, \quad \det(I - \alpha H_a) = \det(I - \alpha H_b) = 1, \tag{3.5}$$

$$\begin{aligned} \operatorname{tr}(K) &= \int_a^b dx \operatorname{tr}_{\mathbb{C}^{n_1}}(g_1(x) f_1(x)) = \int_a^b dx \operatorname{tr}_{\mathbb{C}^m}(f_1(x) g_1(x)) && (3.6) \\ &= \int_a^b dx \operatorname{tr}_{\mathbb{C}^{n_2}}(g_2(x) f_2(x)) = \int_a^b dx \operatorname{tr}_{\mathbb{C}^m}(f_2(x) g_2(x)). && (3.7) \end{aligned}$$

Assume in addition that U is given by (2.38)*. Then,*

$$\begin{aligned} \det(I - \alpha K) &= \det_{\mathbb{C}^{n_1}}(I_{n_1} - \alpha T (I - \alpha H_b)^{-1} S) && (3.8) \\ &= \det_{\mathbb{C}^{n_1}}\bigg(I_{n_1} - \alpha \int_a^b dx\, g_1(x) \hat{f}_1(x, \alpha)\bigg) && (3.9) \\ &= \det_{\mathbb{C}^n}(U(a, \alpha)) && (3.10) \\ &= \det_{\mathbb{C}^{n_2}}(I_{n_2} - \alpha R (I - \alpha H_a)^{-1} Q) && (3.11) \\ &= \det_{\mathbb{C}^{n_2}}\bigg(I_{n_2} - \alpha \int_a^b dx\, g_2(x) \hat{f}_2(x, \alpha)\bigg) && (3.12) \\ &= \det_{\mathbb{C}^n}(U(b, \alpha)). && (3.13) \end{aligned}$$

Proof. We briefly sketch the argument following [11, Theorem 3.2] since we use a different solution U of $U' = \alpha A U$. Relations (3.5) are clear from Lidskii's theorem (cf., e.g., [11, Theorem VII.6.1], [18, Sect. III.8, Sect. IV.1], [36, Theorem 3.7]). Thus,

$$\operatorname{tr}(K) = \operatorname{tr}(QR) = \operatorname{tr}(RQ) = \operatorname{tr}(ST) = \operatorname{tr}(TS) \tag{3.14}$$

then proves (3.6) and (3.7). Next, one observes

$$\begin{aligned} I - \alpha K &= (I - \alpha H_a)[I - \alpha (I - \alpha H_a)^{-1} QR] && (3.15) \\ &= (I - \alpha H_b)[I - \alpha (I - H_b)^{-1} ST] && (3.16) \end{aligned}$$

and hence,

$$
\begin{aligned}
\det(I-\alpha K) &= \det(I-\alpha H_a)\det(I-\alpha(I-\alpha H_a)^{-1}QR) \\
&= \det(I-\alpha(I-\alpha H_a)^{-1}QR) \\
&= \det(I-\alpha R(I-\alpha H_a)^{-1}Q) \\
&= \det_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q) \qquad (3.17)\\
&= \det_{\mathbb{C}^{n}}(U(b,\alpha)). \qquad (3.18)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\det(I-\alpha K) &= \det(I-\alpha H_b)\det(I-\alpha(I-\alpha H_b)^{-1}ST) \\
&= \det(I-\alpha(I-\alpha H_b)^{-1}ST) \\
&= \det(I-\alpha T(I-\alpha H_b)^{-1}S) \\
&= \det_{\mathbb{C}^{n_1}}(I_{n_1}-\alpha T(I-\alpha H_b)^{-1}S) \qquad (3.19)\\
&= \det_{\mathbb{C}^{n}}(U(a,\alpha)). \qquad (3.20)
\end{aligned}
$$

Relations (3.18) and (3.20) follow directly from taking the limit $x \uparrow b$ and $x \downarrow a$ in (2.39). This proves (3.8)–(3.13). $\square$

Equality of (3.18) and (3.20) also follows directly from (2.42) and

$$
\begin{aligned}
\int_a^b dx'\,\mathrm{tr}_{\mathbb{C}^n}(A(x')) &= \int_a^b dx'[\mathrm{tr}_{\mathbb{C}^{n_1}}(g_1(x')f_1(x')) - \mathrm{tr}_{\mathbb{C}^{n_2}}(g_2(x')f_2(x'))] \qquad (3.21)\\
&= \mathrm{tr}(H_a) = \mathrm{tr}(H_b) = 0. \qquad (3.22)
\end{aligned}
$$

Finally, we treat the case of 2-modified Fredholm determinants in the case where K is only assumed to lie in the Hilbert-Schmidt class. In addition to (3.1)–(3.3) we will use the following standard facts for 2-modified Fredholm determinants $\det_2(I-A)$, $A \in \mathcal{B}_2(\mathcal{H})$ (cf. e,g., [13], [14, Ch. XIII], [18, Sect. IV.2], [35], [36, Ch. 3]),

$$\det_2(I-A) = \det((I-A)\exp(A)), \quad A \in \mathcal{B}_2(\mathcal{H}), \tag{3.23}$$

$$\det_2((I-A)(I-B)) = \det_2(I-A)\det_2(I-B)e^{-\mathrm{tr}(AB)}, \quad A, B \in \mathcal{B}_2(\mathcal{H}), \tag{3.24}$$

$$\det_2(I-A) = \det(I-A)e^{\mathrm{tr}(A)}, \quad A \in \mathcal{B}_1(\mathcal{H}). \tag{3.25}$$

Theorem 3.3. *Suppose Hypothesis 2.1 and let $\alpha \in \mathbb{C}$. Then,*

$$\det_2(I-\alpha H_a) = \det_2(I-\alpha H_b) = 1. \tag{3.26}$$

Assume in addition that U is given by (2.38)*. Then,*

$$\begin{aligned}
\det{}_2(I-\alpha K) &= \det_{\mathbb{C}^{n_1}}(I_{n_1}-\alpha T(I-\alpha H_b)^{-1}S)\exp(\alpha\,\mathrm{tr}_{\mathbb{C}^m}(ST)) && (3.27)\\
&= \det_{\mathbb{C}^{n_1}}\bigg(I_{n_1}-\alpha\int_a^b dx\, g_1(x)\hat f_1(x,\alpha)\bigg)\\
&\quad\times\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_1(x)g_1(x))\bigg) && (3.28)\\
&= \det_{\mathbb{C}^n}(U(a,\alpha))\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_1(x)g_1(x))\bigg) && (3.29)\\
&= \det_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q)\exp(\alpha\,\mathrm{tr}_{\mathbb{C}^m}(QR)) && (3.30)\\
&= \det_{\mathbb{C}^{n_2}}\bigg(I_{n_2}-\alpha\int_a^b dx\, g_2(x)\hat f_2(x,\alpha)\bigg)\\
&\quad\times\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_2(x)g_2(x))\bigg) && (3.31)\\
&= \det_{\mathbb{C}^n}(U(b,\alpha))\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_2(x)g_2(x))\bigg). && (3.32)
\end{aligned}$$

Proof. Relations (3.26) follow since the Volterra operators H_a, H_b have no nonzero eigenvalues. Next, again using (3.15) and (3.16), one computes,

$$\begin{aligned}
\det{}_2(I-\alpha K) &= \det{}_2(I-\alpha H_a)\det{}_2(I-\alpha(I-\alpha H_a)^{-1}QR)\\
&\qquad\times\exp(-\mathrm{tr}(\alpha^2 H_a(I-\alpha H_a)^{-1}QR))\\
&= \det(I-\alpha(I-\alpha H_a)^{-1}QR)\exp(\alpha\,\mathrm{tr}((I-\alpha H_a)^{-1}QR))\\
&\quad\times\exp(-\mathrm{tr}(\alpha^2 H_a(I-\alpha H_a)^{-1}QR))\\
&= \det_{\mathbb{C}^{n_2}}(I_{n_2}-\alpha R(I-\alpha H_a)^{-1}Q)\exp(\alpha\,\mathrm{tr}(QR)) && (3.33)\\
&= \det_{\mathbb{C}^n}(U(b,\alpha))\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_1(x)g_1(x))\bigg). && (3.34)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\det{}_2(I-\alpha K) &= \det{}_2(I-\alpha H_b)\det{}_2(I-\alpha(I-\alpha H_b)^{-1}ST)\\
&\quad\times\exp(-\mathrm{tr}(\alpha^2 H_b(I-\alpha H_b)^{-1}ST))\\
&= \det(I-\alpha(I-\alpha H_b)^{-1}ST)\exp(\alpha\,\mathrm{tr}((I-\alpha H_b)^{-1}ST))\\
&\quad\times\exp(-\mathrm{tr}(\alpha^2 H_b(I-\alpha H_b)^{-1}ST))\\
&= \det_{\mathbb{C}^{n_1}}(I_{n_1}-\alpha T(I-\alpha H_b)^{-1}S)\exp(\alpha\,\mathrm{tr}(ST)) && (3.35)\\
&= \det_{\mathbb{C}^n}(U(a,\alpha))\exp\bigg(\alpha\int_a^b dx\,\mathrm{tr}_{\mathbb{C}^m}(f_2(x)g_2(x))\bigg). && (3.36)
\end{aligned}$$

□

Equality of (3.34) and (3.36) also follows directly from (2.42) and (3.21).

4. Some applications to Jost functions, transmission coefficients, and Floquet discriminants of Schrödinger operators

In this section we illustrate the results of Section 3 in three particular cases: The case of Jost functions for half-line Schrödinger operators, the transmission coefficient for Schrödinger operators on the real line, and the case of Floquet discriminants associated with Schrödinger operators on a compact interval. The case of a the second-order Schrödinger operator on the line is also transformed into a first-order 2×2 system and its associated 2-modified Fredholm deteminant is identified with that of the Schrödinger operator on $\mathbb{R}$. For simplicity we will limit ourselves to scalar coefficients although the results for half-line Schrödinger operators and those on the full real line immediately extend to the matrix-valued situation.

We start with the case of half-line Schrödinger operators:
The case $(a,b)=(0,\infty)$: Assuming

$$V\in L^1((0,\infty);dx), \tag{4.1}$$

(we note that V is not necessarily assumed to be real-valued) we introduce the closed Dirichlet-type operators in $L^2((0,\infty);dx)$ defined by

$$\begin{aligned}
&H_+^{(0)}f=-f'',\\
&f\in\operatorname{dom}\big(H_+^{(0)}\big)=\{g\in L^2((0,\infty);dx)\,|\,g,g'\in AC_{\text{loc}}([0,R])\text{ for all }R>0, \\
&\qquad\qquad f(0_+)=0,\ f''\in L^2((0,\infty);dx)\},
\end{aligned}\tag{4.2}$$

$$\begin{aligned}
&H_+f=-f''+Vf,\\
&f\in\operatorname{dom}(H_+)=\{g\in L^2((0,\infty);dx)\,|\,g,g'\in AC_{\text{loc}}([0,R])\text{ for all }R>0, \\
&\qquad\qquad f(0_+)=0,\ (-f''+Vf)\in L^2((0,\infty);dx)\}.
\end{aligned}\tag{4.3}$$

We note that $H_+^{(0)}$ is self-adjoint and that H_+ is self-adjoint if and only if V is real-valued.

Next we introduce the regular solution $\phi(z,\cdot)$ and Jost solution $f(z,\cdot)$ of $-\psi''(z)+V\psi(z)=z\psi(z)$, $z\in\mathbb{C}\backslash\{0\}$, by

$$\phi(z,x)=z^{-1/2}\sin(z^{1/2}x)+\int_0^x dx'g_+^{(0)}(z,x,x')V(x')\phi(z,x'), \tag{4.4}$$

$$f(z,x)=e^{iz^{1/2}x}-\int_x^\infty dx'g_+^{(0)}(z,x,x')V(x')f(z,x'), \tag{4.5}$$

$$\operatorname{Im}(z^{1/2})\geq 0,\ z\neq 0,\ x\geq 0,$$

where

$$g_+^{(0)}(z,x,x')=z^{-1/2}\sin(z^{1/2}(x-x')). \tag{4.6}$$

We also introduce the Green's function of $H_+^{(0)}$,

$$G_+^{(0)}(z,x,x')=\big(H_+^{(0)}-z\big)^{-1}(x,x')=\begin{cases}z^{-1/2}\sin(z^{1/2}x)e^{iz^{1/2}x'}, & x\leq x',\\ z^{-1/2}\sin(z^{1/2}x')e^{iz^{1/2}x}, & x\geq x'.\end{cases} \tag{4.7}$$

The Jost function $\mathcal{F}$ associated with the pair $\big(H_+, H_+^{(0)}\big)$ is given by

$$\begin{aligned}
\mathcal{F}(z) &= W(f(z),\phi(z)) = f(z,0) && (4.8)\\
&= 1 + z^{-1/2}\int_0^\infty dx\, \sin(z^{1/2}x)V(x)f(z,x) && (4.9)\\
&= 1 + \int_0^\infty dx\, e^{iz^{1/2}x}V(x)\phi(z,x); \quad \operatorname{Im}(z^{1/2})\geq 0,\ z\neq 0, && (4.10)
\end{aligned}$$

where

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x), \quad x\geq 0, \tag{4.11}$$

denotes the Wronskian of f and g. Introducing the factorization

$$V(x) = u(x)v(x), \quad u(x) = |V(x)|^{1/2}\exp(i\arg(V(x))),\ v(x) = |V(x)|^{1/2}, \tag{4.12}$$

one verifies[6]

$$\begin{aligned}
(H_+ - z)^{-1} &= \big(H_+^{(0)} - z\big)^{-1}\\
&\quad - \big(H_+^{(0)} - z\big)^{-1} v\Big[I + \overline{u\big(H_+^{(0)} - z\big)^{-1}v}\Big]^{-1} u\big(H_+^{(0)} - z\big)^{-1}, \quad (4.13)\\
&\qquad z\in\mathbb{C}\backslash\operatorname{spec}(H_+).
\end{aligned}$$

To establish the connection with the notation used in Sections 2 and 3, we introduce the operator $K(z)$ in $L^2((0,\infty);dx)$ (cf. (2.3)) by

$$K(z) = -\overline{u\big(H_+^{(0)} - z\big)^{-1}v}, \quad z\in\mathbb{C}\backslash\operatorname{spec}\big(H_+^{(0)}\big) \tag{4.14}$$

with integral kernel

$$K(z,x,x') = -u(x)G_+^{(0)}(z,x,x')v(x'), \quad \operatorname{Im}(z^{1/2})\geq 0,\ x,x'\geq 0, \tag{4.15}$$

and the Volterra operators $H_0(z)$, $H_\infty(z)$ (cf. (2.4), (2.5)) with integral kernel

$$H(z,x,x') = u(x)g_+^{(0)}(z,x,x')v(x'). \tag{4.16}$$

Moreover, we introduce for a.e. $x>0$,

$$\begin{aligned}
f_1(z,x) &= -u(x)e^{iz^{1/2}x}, & g_1(z,x) &= v(x)z^{-1/2}\sin(z^{1/2}x),\\
f_2(z,x) &= -u(x)z^{-1/2}\sin(z^{1/2}x), & g_2(z,x) &= v(x)e^{iz^{1/2}x}.
\end{aligned} \tag{4.17}$$

Assuming temporarily that

$$\operatorname{supp}(V) \text{ is compact} \tag{4.18}$$

in addition to hypothesis (4.1), introducing $\hat{f}_j(z,x)$, $j=1,2$, by

$$\begin{aligned}
\hat{f}_1(z,x) &= f_1(z,x) - \int_x^\infty dx'\, H(z,x,x')\hat{f}_1(z,x'), && (4.19)\\
\hat{f}_2(z,x) &= f_2(z,x) + \int_0^x dx'\, H(z,x,x')\hat{f}_2(z,x'), && (4.20)\\
&\qquad \operatorname{Im}(z^{1/2})\geq 0,\ z\neq 0,\ x\geq 0,
\end{aligned}$$

[6] $\overline{T}$ denotes the operator closure of T and $\operatorname{spec}(\cdot)$ abbreviates the spectrum of a linear operator.

yields solutions $\hat f_j(z,\cdot)\in L^2((0,\infty);dx)$, $j=1,2$. By comparison with (4.4), (4.5), one then identifies

$$\hat f_1(z,x) = -u(x)f(z,x), \tag{4.21}$$

$$\hat f_2(z,x) = -u(x)\phi(z,x). \tag{4.22}$$

We note that the temporary compact support assumption (4.18) on V has only been introduced to guarantee that $f_2(z,\cdot), \hat f_2(z,\cdot) \in L^2((0,\infty);dx)$. This extra hypothesis will soon be removed.

We start with a well-known result.

Theorem 4.1 (Cf., e.g., [33], Theorem XI.20)**.** *Suppose $f,g\in L^q(\mathbb{R};dx)$ for some $2\le q<\infty$. Denote by $f(X)$ the maximally defined multiplication operator by f in $L^2(\mathbb{R};dx)$ and by $g(P)$ the maximal multiplication operator by g in Fourier space[7] $L^2(\mathbb{R};dp)$. Then[8] $f(X)g(P)\in\mathcal{B}_q(L^2(\mathbb{R};dx))$ and*

$$\|f(X)g(P)\|_{\mathcal{B}_q(L^2(\mathbb{R};dx))} \le (2\pi)^{-1/q}\|f\|_{L^q(\mathbb{R};dx)}\|g\|_{L^q(\mathbb{R};dx)}. \tag{4.23}$$

We will use Theorem 4.1, to sketch a proof of the following known result:

Theorem 4.2. *Suppose $V\in L^1((0,\infty);dx)$ and let $z\in\mathbb{C}$ with $\operatorname{Im}(z^{1/2})>0$. Then*

$$K(z)\in\mathcal{B}_1(L^2((0,\infty);dx)). \tag{4.24}$$

Proof. For $z<0$ this is discussed in the proof of [33, Theorem XI.31]. For completeness we briefly sketch the principal arguments of a proof of Theorem 4.2. One possible approach consists of reducing Theorem 4.2 to Theorem 4.1 in the special case $q=2$ by embedding the half-line problem on $(0,\infty)$ into a problem on $\mathbb{R}$ as follows. One introduces the decomposition

$$L^2(\mathbb{R};dx) = L^2((0,\infty);dx)\oplus L^2((-\infty,0);dx), \tag{4.25}$$

and extends u,v,V to $(-\infty,0)$ by putting u,v,V equal to zero on $(-\infty,0)$, introducing

$$\tilde u(x) = \begin{cases} u(x), & x>0,\\ 0, & x<0,\end{cases} \quad \tilde v(x) = \begin{cases} v(x), & x>0,\\ 0, & x<0,\end{cases} \quad \widetilde V(x) = \begin{cases} V(x), & x>0,\\ 0, & x<0.\end{cases} \tag{4.26}$$

Moreover, one considers the Dirichlet Laplace operator $H_D^{(0)}$ in $L^2(\mathbb{R};dx)$ by

$$\begin{aligned} &H_D^{(0)}f = -f'',\\ &\operatorname{dom}\big(H_D^{(0)}\big) = \{g\in L^2(\mathbb{R};dx)\,|\, g,g'\in AC_{\text{loc}}([0,R])\cap AC_{\text{loc}}([-R,0]) \text{ for all } R>0,\\ &\qquad f(0_\pm)=0,\ f''\in L^2(\mathbb{R};dx)\} \end{aligned} \tag{4.27}$$

and introduces

$$\widetilde K(z) = -\overline{\tilde u\big(H_D^{(0)}-z\big)^{-1}\tilde v} = K(z)\oplus 0, \quad \operatorname{Im}(z^{1/2})>0. \tag{4.28}$$

By Krein's formula, the resolvents of the Dirichlet Laplace operator $H_D^{(0)}$ and that of the ordinary Laplacian $H^{(0)}=P^2=-d^2/dx^2$ on $H^{2,2}(\mathbb{R})$ differ precisely by a

[7] That is, $P=-id/dx$ with domain $\operatorname{dom}(P)=H^{2,1}(\mathbb{R})$, the usual Sobolev space.

[8] $\mathcal{B}_q(\mathcal{H})$, $q\ge 1$ denote the usual trace ideals, cf. [18], [36].

rank one operator. Explicitly, one obtains

$$\begin{aligned} G_D^{(0)}(z,x,x') &= G^{(0)}(z,x,x') - G^{(0)}(z,x,0)G^{(0)}(z,0,0)^{-1}G^{(0)}(z,0,x') \\ &= G^{(0)}(z,x,x') - \frac{i}{2z^{1/2}}\exp(iz^{1/2}|x|)\exp(iz^{1/2}|x'|), \\ &\qquad \operatorname{Im}(z^{1/2})>0,\ x,x'\in\mathbb{R}, \end{aligned} \tag{4.29}$$

where we abbreviated the Green's functions of $H_D^{(0)}$ and $H^{(0)} = -d^2/dx^2$ by

$$G_D^{(0)}(z,x,x') = \big(H_D^{(0)} - z\big)^{-1}(x,x'), \tag{4.30}$$

$$G^{(0)}(z,x,x') = \big(H^{(0)} - z\big)^{-1}(x,x') = \frac{i}{2z^{1/2}}\exp(iz^{1/2}|x-x'|). \tag{4.31}$$

Thus,

$$\widetilde{K}(z) = -\overline{\tilde{u}\big(H^{(0)}-z\big)^{-1}\tilde{v}} - \frac{i}{2z^{1/2}}\big(\tilde{v}\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde{u}\exp(iz^{1/2}|\cdot|). \tag{4.32}$$

By Theorem 4.1 for $q=2$ one infers that

$$\big[\tilde{u}\big(H^{(0)}-z\big)^{-1/2}\big] \in \mathcal{B}_2(L^2(\mathbb{R};dx)), \quad \operatorname{Im}(z^{1/2})>0 \tag{4.33}$$

and hence,

$$\big[\tilde{u}\big(H^{(0)}-z\big)^{-1/2}\big]\overline{\big[\big(H^{(0)}-z\big)^{-1/2}\tilde{v}\big]} \in \mathcal{B}_1(L^2(\mathbb{R};dx)), \quad \operatorname{Im}(z^{1/2})>0. \tag{4.34}$$

Since the second term on the right-hand side of (4.32) is a rank one operator one concludes

$$\widetilde{K}(z) \in \mathcal{B}_1(L^2(\mathbb{R};dx)), \quad \operatorname{Im}(z^{1/2})>0 \tag{4.35}$$

and hence (4.24) using (4.28). □

An application of Lemma 2.6 and Theorem 3.2 then yields the following well-known result identifying the Fredholm determinant of $I-K(z)$ and the Jost function $\mathcal{F}(z)$.

Theorem 4.3. *Suppose $V\in L^1((0,\infty);dx)$ and let $z\in\mathbb{C}$ with $\operatorname{Im}(z^{1/2})>0$. Then*

$$\det(I-K(z)) = \mathcal{F}(z). \tag{4.36}$$

Proof. Assuming temporarily that $\operatorname{supp}(V)$ is compact (cf. (4.18)), Lemma 2.6 applies and one obtains from (2.38) and (4.17)–(4.22) that

$$\begin{aligned} U(z,x) &= \begin{pmatrix} 1-\int_x^\infty dx'\, g_1(z,x')\hat{f}_1(z,x') & \int_0^x dx'\, g_1(z,x')\hat{f}_2(z,x') \\ \int_x^\infty dx'\, g_2(z,x')\hat{f}_1(z,x') & 1-\int_0^x dx'\, g_2(z,x')\hat{f}_2(z,x') \end{pmatrix}, \\ &= \begin{pmatrix} 1+\int_x^\infty dx'\, \frac{\sin(z^{1/2}x')}{z^{1/2}}V(x')f(z,x') & -\int_0^x dx'\, \frac{\sin(z^{1/2}x')}{z^{1/2}}V(x')\phi(z,x') \\ -\int_x^\infty dx'\, e^{iz^{1/2}x'}V(x')f(z,x') & 1+\int_0^x dx'\, e^{iz^{1/2}x'}V(x')\phi(z,x') \end{pmatrix}, \\ &\qquad x>0. \end{aligned} \tag{4.37}$$

Relations (3.9) and (3.12) of Theorem 3.2 with $m = n_1 = n_2 = 1$, $n = 2$, then immediately yield

$$\begin{aligned} \det(I - K(z)) &= 1 + z^{-1/2}\int_0^\infty dx\, \sin(z^{1/2}x)V(x)f(z,x) \\ &= 1 + \int_0^\infty dx\, e^{iz^{1/2}x}V(x)\phi(z,x) \\ &= \mathcal{F}(z) \end{aligned} \tag{4.38}$$

and hence (4.36) is proved under the additional hypothesis (4.18). Removing the compact support hypothesis on V now follows by a standard argument. For completeness we sketch this argument next. Multiplying u, v, V by a smooth cutoff function χ_ε of compact support of the type

$$0 \le \chi \le 1, \quad \chi(x) = \begin{cases} 1, & x \in [0,1], \\ 0, & |x| \ge 2, \end{cases} \quad \chi_\varepsilon(x) = \chi(\varepsilon x),\ \varepsilon > 0, \tag{4.39}$$

denoting the results by $u_\varepsilon = u\chi_\varepsilon$, $v_\varepsilon = v\chi_\varepsilon$, $V_\varepsilon = V\chi_\varepsilon$, one introduces in analogy to (4.26),

$$\tilde u_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x > 0, \\ 0, & x < 0, \end{cases} \quad \tilde v_\varepsilon(x) = \begin{cases} v_\varepsilon(x), & x > 0, \\ 0, & x < 0, \end{cases} \quad \tilde V_\varepsilon(x) = \begin{cases} V_\varepsilon(x), & x > 0, \\ 0, & x < 0, \end{cases} \tag{4.40}$$

and similarly, in analogy to (4.14) and (4.28),

$$K_\varepsilon(z) = -\overline{u_\varepsilon\big(H_+^{(0)} - z\big)^{-1} v_\varepsilon}, \quad \operatorname{Im}(z^{1/2}) > 0, \tag{4.41}$$

$$\widetilde K_\varepsilon(z) = -\overline{\tilde u_\varepsilon\big(H_D^{(0)} - z\big)^{-1} \tilde v_\varepsilon} = K_\varepsilon(z) \oplus 0, \quad \operatorname{Im}(z^{1/2}) > 0. \tag{4.42}$$

One then estimates,

$$\begin{aligned} &\big\|\widetilde K(z) - \widetilde K_\varepsilon(z)\big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad\le \Big\| -\overline{\tilde u\big(H^{(0)} - z\big)^{-1}\tilde v} + \overline{\tilde u_\varepsilon\big(H^{(0)} - z\big)^{-1}\tilde v_\varepsilon}\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\qquad + \frac{1}{2|z|^{1/2}}\Big\|\big(\tilde v\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u\exp(iz^{1/2}|\cdot|) \\ &\qquad\qquad - \big(\tilde v_\varepsilon\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u_\varepsilon\exp(iz^{1/2}|\cdot|)\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\quad\le \Big\| -\overline{\tilde u\big(H^{(0)} - z\big)^{-1}\tilde v} + \overline{\tilde u_\varepsilon\big(H^{(0)} - z\big)^{-1}\tilde v} \\ &\qquad - \overline{\tilde u_\varepsilon\big(H^{(0)} - z\big)^{-1}\tilde v} + \overline{\tilde u_\varepsilon\big(H^{(0)} - z\big)^{-1}\tilde v_\varepsilon}\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \\ &\qquad + \frac{1}{2|z|^{1/2}}\Big\|\big(\tilde v\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u\exp(iz^{1/2}|\cdot|) \\ &\qquad\qquad - \big(\tilde v\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u_\varepsilon\exp(iz^{1/2}|\cdot|) \\ &\qquad\qquad + \big(\tilde v\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u_\varepsilon\exp(iz^{1/2}|\cdot|) \\ &\qquad\qquad - \big(\tilde v_\varepsilon\,\overline{\exp(iz^{1/2}|\cdot|)}\,,\,\cdot\,\big)\tilde u_\varepsilon\exp(iz^{1/2}|\cdot|)\Big\|_{\mathcal{B}_1(L^2(\mathbb{R};dx))} \end{aligned}$$

$$\leq \widetilde{C}(z)\big[\|\tilde{u}-\tilde{u}_\varepsilon\|_{L^2(\mathbb{R};dx)}+\|\tilde{v}-\tilde{v}_\varepsilon\|_{L^2(\mathbb{R};dx)}\big]=C(z)\|\tilde{v}-\tilde{v}_\varepsilon\|_{L^2(\mathbb{R};dx)}$$
$$\leq C(z)\|v-v_\varepsilon\|_{L^2((0,\infty);dx)}, \tag{4.43}$$

where $C(z)=2\widetilde{C}(z)>0$ is an appropriate constant. Thus, applying (4.28) and (4.42), one finally concludes

$$\lim_{\varepsilon\downarrow 0}\big\|K(z)-K_\varepsilon(z)\big\|_{\mathcal{B}_1(L^2((0,\infty);dx))}=0. \tag{4.44}$$

Since V_ε has compact support, (4.38) applies to V_ε and one obtains,

$$\det(I-K_\varepsilon(z))=\mathcal{F}_\varepsilon(z), \tag{4.45}$$

where, in obvious notation, we add the subscript ε to all quantities associated with V_ε resulting in ϕ_ε, f_ε, $\mathcal{F}_\varepsilon$, $f_{\varepsilon,j}$, $\hat{f}_{\varepsilon,j}$, $j=1,2$, etc. By (4.44), the left-hand side of (4.45) converges to $\det(I-K(z))$ as $\varepsilon\downarrow 0$. Since

$$\lim_{\varepsilon\downarrow 0}\|V_\varepsilon-V\|_{L^1((0,\infty);dx)}=0, \tag{4.46}$$

the Jost function $\mathcal{F}_\varepsilon$ is well-known to converge to $\mathcal{F}$ pointwise as $\varepsilon\downarrow 0$ (cf. [5]). Indeed, fixing z and iterating the Volterra integral equation (4.5) for f_ε shows that $|z^{-1/2}\sin(z^{1/2}x)f_\varepsilon(z,x)|$ is uniformly bounded with respect to (x,ε) and hence the continuity of $\mathcal{F}_\varepsilon(z)$ with respect to ε follows from (4.46) and the analog of (4.9) for V_ε,

$$\mathcal{F}_\varepsilon(z)=1+z^{-1/2}\int_0^\infty dx\,\sin(z^{1/2}x)V_\varepsilon(x)f_\varepsilon(z,x), \tag{4.47}$$

applying the dominated convergence theorem. Hence, (4.45) yields (4.36) in the limit $\varepsilon\downarrow 0$. $\square$

Remark 4.4. (i) *The result* (4.38) *explicitly shows that* $\det_{\mathbb{C}^n}(U(z,0))$ *vanishes for each eigenvalue* z (*one then necessarily has* $z<0$) *of the Schrödinger operator* H*. Hence, a normalization of the type* $U(z,0)=I_n$ *is clearly impossible in such a case.*
(ii) *The right-hand side* $\mathcal{F}$ *of* (4.36) (*and hence the Fredholm determinant on the left-hand side*) *admits a continuous extension to the positive real line. Imposing the additional exponential falloff of the potential of the type* $V\in L^1((0,\infty);\exp(ax)dx)$ *for some* $a>0$, *then* $\mathcal{F}$ *and hence the Fredholm determinant on the left-hand side of* (4.36) *permit an analytic continuation through the essential spectrum of* H_+ *into a strip of width* $a/2$ (*w.r.t. the variable* $z^{1/2}$)*. This is of particular relevance in the study of resonances of* H_+ (*cf.* [37]).

The result (4.36) is well-known, we refer, for instance, to [23], [29], [30], [32, p. 344–345], [37]. (Strictly speaking, these authors additionally assume V to be real-valued, but this is not essential in this context.) The current derivation presented appears to be by far the simplest available in the literature as it only involves the elementary manipulations leading to (3.8)–(3.13), followed by a standard approximation argument to remove the compact support hypothesis on V.

Since one is dealing with the Dirichlet Laplacian on $(0,\infty)$ in the half-line context, Theorem 4.2 extends to a larger potential class characterized by

$$\int_0^R dx\,x|V(x)|+\int_R^\infty dx\,|V(x)|<\infty \tag{4.48}$$

for some fixed $R > 0$. We omit the corresponding details but refer to [33, Theorem XI.31], which contains the necessary basic facts to make the transition from hypothesis (4.1) to (4.48).

Next we turn to Schrödinger operators on the real line:
The case $(a,b) = \mathbb{R}$: Assuming

$$V \in L^1(\mathbb{R}; dx), \tag{4.49}$$

we introduce the closed operators in $L^2(\mathbb{R}; dx)$ defined by

$$H^{(0)} f = -f'', \quad f \in \mathrm{dom}\big(H^{(0)}\big) = H^{2,2}(\mathbb{R}), \tag{4.50}$$

$$Hf = -f'' + Vf, \tag{4.51}$$

$$f \in \mathrm{dom}(H) = \{g \in L^2(\mathbb{R}; dx) \,|\, g, g' \in AC_{\mathrm{loc}}(\mathbb{R});\, (-f'' + Vf) \in L^2(\mathbb{R}); dx)\}.$$

Again, $H^{(0)}$ is self-adjoint. Moreover, H is self-adjoint if and only if V is real-valued.

Next we introduce the Jost solutions $f_\pm(z,\cdot)$ of $-\psi''(z) + V\psi(z) = z\psi(z)$, $z \in \mathbb{C}\backslash\{0\}$, by

$$f_\pm(z,x) = e^{\pm iz^{1/2}x} - \int_x^{\pm\infty} dx'\, g^{(0)}(z,x,x')V(x')f_\pm(z,x'), \tag{4.52}$$
$$\mathrm{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R},$$

where $g^{(0)}(z,x,x')$ is still given by (4.6). We also introduce the Green's function of $H^{(0)}$,

$$G^{(0)}(z,x,x') = \big(H^{(0)} - z\big)^{-1}(x,x') = \frac{i}{2z^{1/2}} e^{iz^{1/2}|x-x'|}, \quad \mathrm{Im}(z^{1/2}) > 0,\ x,x' \in \mathbb{R}. \tag{4.53}$$

The Jost function $\mathcal{F}$ associated with the pair $\big(H, H^{(0)}\big)$ is given by

$$\mathcal{F}(z) = \frac{W(f_-(z), f_+(z))}{2iz^{1/2}} \tag{4.54}$$

$$= 1 - \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} dx\, e^{\mp iz^{1/2}x} V(x) f_\pm(z,x), \quad \mathrm{Im}(z^{1/2}) \geq 0,\ z \neq 0, \tag{4.55}$$

where $W(\cdot,\cdot)$ denotes the Wronskian defined in (4.11). We note that if $H^{(0)}$ and H are self-adjoint, then

$$T(\lambda) = \lim_{\varepsilon \downarrow 0} \mathcal{F}(\lambda + i\varepsilon)^{-1}, \quad \lambda > 0, \tag{4.56}$$

denotes the transmission coefficient corresponding to the pair $\big(H, H^{(0)}\big)$. Introducing again the factorization (4.12) of $V = uv$, one verifies as in (4.13) that

$$\begin{aligned}(H - z)^{-1} &= \big(H^{(0)} - z\big)^{-1} \\ &\quad - \big(H^{(0)} - z\big)^{-1} v \Big[I + \overline{u\big(H^{(0)} - z\big)^{-1} v}\Big]^{-1} u \big(H^{(0)} - z\big)^{-1}, \\ &\quad z \in \mathbb{C}\backslash\mathrm{spec}(H).\end{aligned} \tag{4.57}$$

To make contact with the notation used in Sections 2 and 3, we introduce the operator $K(z)$ in $L^2(\mathbb{R}; dx)$ (cf. (2.3), (4.14)) by

$$K(z) = -\overline{u\big(H^{(0)} - z\big)^{-1} v}, \quad z \in \mathbb{C}\backslash\mathrm{spec}\big(H^{(0)}\big) \tag{4.58}$$

with integral kernel

$$K(z,x,x') = -u(x)G^{(0)}(z,x,x')v(x'), \quad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x,x' \in \mathbb{R}, \tag{4.59}$$

and the Volterra operators $H_{-\infty}(z)$, $H_{\infty}(z)$ (cf. (2.4), (2.5)) with integral kernel

$$H(z,x,x') = u(x)g^{(0)}(z,x,x')v(x'). \tag{4.60}$$

Moreover, we introduce for a.e. $x \in \mathbb{R}$,

$$\begin{aligned} f_1(z,x) &= -u(x)e^{iz^{1/2}x}, \qquad g_1(z,x) = (i/2)z^{-1/2}v(x)e^{-iz^{1/2}x}, \\ f_2(z,x) &= -u(x)e^{-iz^{1/2}x}, \quad\ \ g_2(z,x) = (i/2)z^{-1/2}v(x)e^{iz^{1/2}x}. \end{aligned} \tag{4.61}$$

Assuming temporarily that

$$\operatorname{supp}(V) \text{ is compact} \tag{4.62}$$

in addition to hypothesis (4.49), introducing $\hat{f}_j(z,x)$, $j=1,2$, by

$$\hat{f}_1(z,x) = f_1(z,x) - \int_x^\infty dx'\, H(z,x,x')\hat{f}_1(z,x'), \tag{4.63}$$

$$\hat{f}_2(z,x) = f_2(z,x) + \int_{-\infty}^x dx'\, H(z,x,x')\hat{f}_2(z,x'), \tag{4.64}$$

$$\operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R},$$

yields solutions $\hat{f}_j(z,\cdot) \in L^2(\mathbb{R};dx)$, $j=1,2$. By comparison with (4.52), one then identifies

$$\hat{f}_1(z,x) = -u(x)f_+(z,x), \tag{4.65}$$

$$\hat{f}_2(z,x) = -u(x)f_-(z,x). \tag{4.66}$$

We note that the temporary compact support assumption (4.18) on V has only been introduced to guarantee that $f_j(z,\cdot), \hat{f}_j(z,\cdot) \in L^2(\mathbb{R};dx)$, $j=1,2$. This extra hypothesis will soon be removed.

We also recall the well-known result.

Theorem 4.5. *Suppose $V \in L^1(\mathbb{R};dx)$ and let $z \in \mathbb{C}$ with $\operatorname{Im}(z^{1/2}) > 0$. Then*

$$K(z) \in \mathcal{B}_1(L^2(\mathbb{R};dx)). \tag{4.67}$$

This is an immediate consequence of Theorem 4.1 with $q=2$.

An application of Lemma 2.6 and Theorem 3.2 then again yields the following well-known result identifying the Fredholm determinant of $I-K(z)$ and the Jost function $\mathcal{F}(z)$ (inverse transmission coefficient).

Theorem 4.6. *Suppose $V \in L^1(\mathbb{R};dx)$ and let $z \in \mathbb{C}$ with $\operatorname{Im}(z^{1/2}) > 0$. Then*

$$\det(I-K(z)) = \mathcal{F}(z). \tag{4.68}$$

Proof. Assuming temporarily that $\operatorname{supp}(V)$ is compact (cf. (4.18)), Lemma 2.6 applies and one infers from (2.38) and (4.61)–(4.66) that

$$U(z,x) = \begin{pmatrix} 1 - \int_x^\infty dx'\, g_1(z,x')\hat{f}_1(z,x') & \int_{-\infty}^x dx'\, g_1(z,x')\hat{f}_2(z,x') \\ \int_x^\infty dx'\, g_2(z,x')\hat{f}_1(z,x') & 1 - \int_{-\infty}^x dx'\, g_2(z,x')\hat{f}_2(z,x') \end{pmatrix}, \quad x \in \mathbb{R}, \tag{4.69}$$

becomes

$$U_{1,1}(z,x) = 1 + \frac{i}{2z^{1/2}} \int_x^\infty dx'\, e^{-iz^{1/2}x'} V(x') f_+(z,x'), \tag{4.70}$$

$$U_{1,2}(z,x) = -\frac{i}{2z^{1/2}} \int_{-\infty}^x dx'\, e^{-iz^{1/2}x'} V(x') f_-(z,x'), \tag{4.71}$$

$$U_{2,1}(z,x) = -\frac{i}{2z^{1/2}} \int_x^\infty dx'\, e^{iz^{1/2}x'} V(x') f_+(z,x'), \tag{4.72}$$

$$U_{2,2}(z,x) = 1 + \frac{i}{2z^{1/2}} \int_{-\infty}^x dx'\, e^{iz^{1/2}x'} V(x') f_-(z,x'). \tag{4.73}$$

Relations (3.9) and (3.12) of Theorem 3.2 with $m = n_1 = n_2 = 1$, $n = 2$, then immediately yield

$$\begin{aligned} \det(I - K(z)) &= 1 - \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} dx\, e^{\mp iz^{1/2}x} V(x) f_\pm(z,x) \\ &= \mathcal{F}(z) \end{aligned} \tag{4.74}$$

and hence (4.68) is proved under the additional hypothesis (4.62). Removing the compact support hypothesis on V now follows line by line the approximation argument discussed in the proof of Theorem 4.3. $\square$

Remark 4.4 applies again to the present case of Schrödinger operators on the line. In particular, if one imposes the additional exponential falloff of the potential V of the type $V \in L^1(\mathbb{R}; \exp(a|x|)dx)$ for some $a > 0$, then $\mathcal{F}$ and hence the Fredholm determinant on the left-hand side of (4.68) permit an analytic continuation through the essential spectrum of H into a strip of width $a/2$ (w.r.t. the variable $z^{1/2}$). This is of relevance to the study of resonances of H (cf., e.g., [8], [37], and the literature cited therein).

The result (4.68) is well-known (although, typically under the additional assumption that V be real-valued), see, for instance, [9], [31, Appendix A], [36, Proposition 5.7], [37]. Again, the derivation just presented appears to be the most streamlined available for the reasons outlined after Remark 4.4.

For an explicit expansion of Fredholm determinants of the type (4.15) and (4.59) (valid in the case of general Green's functions G of Schrödinger operators H, not just for $G^{(0)}$ associated with $H^{(0)}$) we refer to Proposition 2.8 in [35].

Next, we revisit the result (4.68) from a different and perhaps somewhat unusual perspective. We intend to rederive the analogous result in the context of 2-modified determinants $\det_2(\cdot)$ by rewriting the scalar second-order Schrödinger equation as a first-order 2×2 system, taking the latter as our point of departure.

Assuming hypothesis 4.49 for the rest of this example, the Schrödinger equation

$$-\psi''(z,x) + V(x)\psi(z,x) = z\psi(z,x), \tag{4.75}$$

is equivalent to the first-order system

$$\Psi'(z,x) = \begin{pmatrix} 0 & 1 \\ V(x) - z & 0 \end{pmatrix} \Psi(z,x), \quad \Psi(z,x) = \begin{pmatrix} \psi(z,x) \\ \psi'(z,x) \end{pmatrix}. \tag{4.76}$$

Since $\Phi^{(0)}$ defined by

$$\Phi^{(0)}(z,x) = \begin{pmatrix} \exp(-iz^{1/2}x) & \exp(iz^{1/2}x) \\ -iz^{1/2}\exp(-iz^{1/2}x) & iz^{1/2}\exp(iz^{1/2}x) \end{pmatrix}, \quad \operatorname{Im}(z^{1/2}) \geq 0 \tag{4.77}$$

with

$$\det_{\mathbb{C}^2}(\Phi^{(0)}(z,x)) = 1, \quad (z,x) \in \mathbb{C} \times \mathbb{R}, \tag{4.78}$$

is a fundamental matrix of the system (4.76) in the case $V = 0$ a.e., and since

$$\Phi^{(0)}(z,x)\Phi^{(0)}(z,x')^{-1} = \begin{pmatrix} \cos(z^{1/2}(x-x')) & z^{-1/2}\sin(z^{1/2}(x-x')) \\ -z^{1/2}\sin(z^{1/2}(x-x')) & \cos(z^{1/2}(x-x')) \end{pmatrix}, \tag{4.79}$$

the system (4.76) has the following pair of linearly independent solutions for $z \neq 0$,

$$\begin{aligned} F_\pm(z,x) &= F_\pm^{(0)}(z,x) \\ &\quad - \int_x^{\pm\infty} dx' \begin{pmatrix} \cos(z^{1/2}(x-x')) & z^{-1/2}\sin(z^{1/2}(x-x')) \\ -z^{1/2}\sin(z^{1/2}(x-x')) & \cos(z^{1/2}(x-x')) \end{pmatrix} \\ &\qquad\qquad \times \begin{pmatrix} 0 & 0 \\ V(x') & 0 \end{pmatrix} F_\pm(z,x') \\ &= F_\pm^{(0)}(z,x) - \int_x^{\pm\infty} dx' \begin{pmatrix} z^{-1/2}\sin(z^{1/2}(x-x')) & 0 \\ \cos(z^{1/2}(x-x')) & 0 \end{pmatrix} V(x')F_\pm(z,x'), \\ &\qquad\qquad\qquad\qquad\qquad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R}, \end{aligned} \tag{4.80}$$

where we abbreviated

$$F_\pm^{(0)}(z,x) = \begin{pmatrix} 1 \\ \pm iz^{1/2} \end{pmatrix} \exp(\pm iz^{1/2}x). \tag{4.81}$$

By inspection, the first component of (4.80) is equivalent to (4.52) and the second component to the x-derivative of (4.52), that is, one has

$$F_\pm(z,,x) = \begin{pmatrix} f_\pm(z,x) \\ f'_\pm(z,x) \end{pmatrix}, \quad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R}. \tag{4.82}$$

Next, one introduces

$$\begin{aligned} f_1(z,x) &= -u(x) \begin{pmatrix} 1 \\ iz^{1/2} \end{pmatrix} \exp(iz^{1/2}x), \\ f_2(z,x) &= -u(x) \begin{pmatrix} 1 \\ -iz^{1/2} \end{pmatrix} \exp(-iz^{1/2}x), \\ g_1(z,x) &= v(x) \begin{pmatrix} \frac{i}{2z^{1/2}} \exp(-iz^{1/2}x) & 0 \end{pmatrix}, \\ g_2(z,x) &= v(x) \begin{pmatrix} \frac{i}{2z^{1/2}} \exp(iz^{1/2}x) & 0 \end{pmatrix} \end{aligned} \tag{4.83}$$

and hence

$$H(z,x,x') = f_1(z,x)g_1(z,x') - f_2(z,x)g_2(z,x') \tag{4.84}$$

$$= u(x) \begin{pmatrix} z^{-1/2}\sin(z^{1/2}(x-x')) & 0 \\ \cos(z^{1/2}(x-x')) & 0 \end{pmatrix} v(x') \tag{4.85}$$

and we introduce

$$\widetilde{K}(z,x,x') = \begin{cases} f_1(z,x)g_1(z,x'), & x' < x, \\ f_2(z,x)g_2(z,x'), & x < x', \end{cases} \tag{4.86}$$

$$= \begin{cases} -u(x)\frac{1}{2}\exp(iz^{1/2}(x-x'))\begin{pmatrix} iz^{-1/2} & 0 \\ -1 & 0 \end{pmatrix} v(x'), & x' < x, \\ -u(x)\frac{1}{2}\exp(-iz^{1/2}(x-x'))\begin{pmatrix} iz^{-1/2} & 0 \\ 1 & 0 \end{pmatrix} v(x'), & x < x', \end{cases} \tag{4.87}$$

$$\operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x, x' \in \mathbb{R}.$$

We note that $\widetilde{K}(z,\cdot,\cdot)$ is discontinuous on the diagonal $x = x'$. Since

$$\widetilde{K}(z,\cdot,\cdot) \in L^2(\mathbb{R}^2; dx\,dx'), \quad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0, \tag{4.88}$$

the associated operator $\widetilde{K}(z)$ with integral kernel (4.87) is Hilbert–Schmidt,

$$\widetilde{K}(z) \in \mathcal{B}_2(L^2(\mathbb{R}; dx)), \quad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0. \tag{4.89}$$

Next, assuming temporarily that

$$\operatorname{supp}(V) \text{ is compact}, \tag{4.90}$$

the integral equations defining $\hat{f}_j(z,x)$, $j = 1, 2$,

$$\hat{f}_1(z,x) = f_1(z,x) - \int_x^\infty dx'\, H(z,x,x')\hat{f}_1(z,x'), \tag{4.91}$$

$$\hat{f}_2(z,x) = f_2(z,x) + \int_{-\infty}^x dx'\, H(z,x,x')\hat{f}_2(z,x'), \tag{4.92}$$

$$\operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R},$$

yield solutions $\hat{f}_j(z,\cdot) \in L^2(\mathbb{R}; dx)$, $j = 1, 2$. By comparison with (4.80), one then identifies

$$\hat{f}_1(z,x) = -u(x)F_+(z,x), \tag{4.93}$$

$$\hat{f}_2(z,x) = -u(x)F_-(z,x). \tag{4.94}$$

We note that the temporary compact support assumption (4.90) on V has only been introduced to guarantee that $f_j(z,\cdot), \hat{f}_j(z,\cdot) \in L^2(\mathbb{R}; dx)^2$, $j = 1, 2$. This extra hypothesis will soon be removed.

An application of Lemma 2.6 and Theorem 3.3 then yields the following result.

Theorem 4.7. *Suppose $V \in L^1(\mathbb{R}; dx)$ and let $z \in \mathbb{C}$ with $\operatorname{Im}(z^{1/2}) \geq 0$, $z \neq 0$. Then*

$$\det{}_2(I - \widetilde{K}(z)) = \mathcal{F}(z)\exp\left(-\frac{i}{2z^{1/2}}\int_{\mathbb{R}} dx\, V(x)\right) \tag{4.95}$$

$$= \det{}_2(I - K(z)) \tag{4.96}$$

with $K(z)$ defined in (4.58).

Proof. Assuming temporarily that $\operatorname{supp}(V)$ is compact (cf. (4.90)) equation (4.95) directly follows from combining (3.28) (or (3.31)) with $a = -\infty$, $b = \infty$, (3.17) (or (3.19)), (4.68), and (4.83). Equation (4.96) then follows from (3.25), (3.6) (or

(3.7)), and (4.83). To extend the result to general $V \in L^1(\mathbb{R}; dx)$ one follows the approximation argument presented in Theorem 4.3. $\square$

One concludes that the scalar second-order equation (4.75) and the first-order system (4.76) share the identical 2-modified Fredholm determinant.

Remark 4.8. *Let* $\operatorname{Im}(z^{1/2}) \geq 0$, $z \neq 0$, *and* $x \in \mathbb{R}$. *Then following up on Remark 2.8, one computes*

$$\begin{aligned} A(z,x) &= \begin{pmatrix} g_1(z,x)f_1(z,x) & g_1(z,x)f_2(z,x) \\ -g_2(z,x)f_1(z,x) & -g_2(z,x)f_2(z,x) \end{pmatrix} \\ &= -\frac{i}{2z^{1/2}}V(x)\begin{pmatrix} 1 & e^{-2iz^{1/2}x} \\ -e^{2iz^{1/2}x} & -1 \end{pmatrix} \\ &= -\frac{i}{2z^{1/2}}V(x)\begin{pmatrix} e^{-iz^{1/2}x} & 0 \\ 0 & e^{iz^{1/2}x} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^{iz^{1/2}x} & 0 \\ 0 & e^{-iz^{1/2}x} \end{pmatrix}. \end{aligned} \tag{4.97}$$

Introducing

$$W(z,x) = e^{M(z)x}U(z,x), \quad M(z) = iz^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.98}$$

and recalling

$$U'(z,x) = A(z,x)U(z,x), \tag{4.99}$$

(*cf.* (2.20)), *equation* (4.99) *reduces to*

$$W'(z,x) = \left[iz^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{i}{2z^{1/2}}V(x)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right]W(z,x). \tag{4.100}$$

Moreover, introducing

$$T(z) = \begin{pmatrix} 1 & 1 \\ iz^{1/2} & -iz^{1/2} \end{pmatrix}, \quad \operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0, \tag{4.101}$$

one obtains

$$\begin{aligned} \left[iz^{1/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{i}{2z^{1/2}}V(x)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right] &= T(z)^{-1}\begin{pmatrix} 0 & 1 \\ V(x) - z & 0 \end{pmatrix}T(z), \\ &\operatorname{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \in \mathbb{R}, \end{aligned} \tag{4.102}$$

which demonstrates the connection between (2.20), (4.100), *and* (4.76).

Finally, we turn to the case of periodic Schrödinger operators of period $\omega > 0$:
The case $(\mathbf{a}, \mathbf{b}) = (\mathbf{0}, \omega)$: Assuming

$$V \in L^1((0,\omega); dx), \tag{4.103}$$

we introduce two one-parameter families of closed operators in $L^2((0,\omega);dx)$ defined by

$$\begin{aligned}
&H_\theta^{(0)} f = -f'', \\
&f \in \operatorname{dom}\big(H_\theta^{(0)}\big) = \{g \in L^2((0,\omega);dx) \,|\, g, g' \in AC([0,\omega]);\; g(\omega) = e^{i\theta} g(0), \\
&\qquad\qquad g'(\omega) = e^{i\theta} g'(0),\; g'' \in L^2((0,\omega);dx)\}, \qquad (4.104) \\
&H_\theta f = -f'' + Vf, \\
&f \in \operatorname{dom}(H_\theta) = \{g \in L^2((0,\omega);dx) \,|\, g, g' \in AC([0,\omega]);\; g(\omega) = e^{i\theta} g(0), \\
&\qquad\qquad g'(\omega) = e^{i\theta} g'(0),\; (-g'' + Vg) \in L^2((0,\omega);dx)\}, \qquad (4.105)
\end{aligned}$$

where $\theta \in [0, 2\pi)$. As in the previous cases considered, $H_\theta^{(0)}$ is self-adjoint and H_θ is self-adjoint if and only if V is real-valued.

Introducing the fundamental system of solutions $c(z,\cdot)$ and $s(z,\cdot)$ of $-\psi''(z) + V\psi(z) = z\psi(z)$, $z \in \mathbb{C}$, by

$$c(z,0) = 1 = s'(z,0), \quad c'(z,0) = 0 = s(z,0), \tag{4.106}$$

the associated fundamental matrix of solutions $\Phi(z,x)$ is defined by

$$\Phi(z,x) = \begin{pmatrix} c(z,x) & s(z,x) \\ c'(z,x) & s'(z,x) \end{pmatrix}. \tag{4.107}$$

The monodromy matrix is then given by $\Phi(z,\omega)$, and the Floquet discriminant $\Delta(z)$ is defined as half of the trace of the latter,

$$\Delta(z) = \operatorname{tr}_{\mathbb{C}^2}(\Phi(z,\omega))/2 = [c(z,\omega) + s'(z,\omega)]/2. \tag{4.108}$$

Thus, the eigenvalue equation for H_θ reads,

$$\Delta(z) = \cos(\theta). \tag{4.109}$$

In the special case $V = 0$ a.e. one obtains

$$c^{(0)}(z,x) = \cos(z^{1/2}x), \quad s^{(0)}(z,x) = \sin(z^{1/2}x) \tag{4.110}$$

and hence,

$$\Delta^{(0)}(z) = \cos(z^{1/2}\omega). \tag{4.111}$$

Next we introduce additional solutions $\varphi_\pm(z,\cdot)$, $\psi_\pm(z,\cdot)$ of $-\psi''(z) + V\psi(z) = z\psi(z)$, $z \in \mathbb{C}$, by

$$\begin{aligned}
\varphi_\pm(z,x) &= e^{\pm iz^{1/2}x} + \int_0^x dx'\, g^{(0)}(z,x,x')V(x')\varphi_\pm(z,x'), \qquad (4.112) \\
\psi_\pm(z,x) &= e^{\pm iz^{1/2}x} - \int_x^\omega dx'\, g^{(0)}(z,x,x')V(x')\psi_\pm(z,x'), \qquad (4.113) \\
&\qquad\qquad \operatorname{Im}(z^{1/2}) \geq 0,\; x \in [0,\omega],
\end{aligned}$$

where $g^{(0)}(z,x,x')$ is still given by (4.6). We also introduce the Green's function of $H_\theta^{(0)}$,

$$\begin{aligned}
G_\theta^{(0)}(z,x,x') &= \big(H_\theta^{(0)} - z\big)^{-1}(x,x') \\
&= \frac{i}{2z^{1/2}}\bigg[e^{iz^{1/2}|x-x'|} + \frac{e^{iz^{1/2}(x-x')}}{e^{i\theta}e^{-iz^{1/2}\omega} - 1} + \frac{e^{-iz^{1/2}(x-x')}}{e^{-i\theta}e^{-iz^{1/2}\omega} - 1}\bigg], \qquad (4.114) \\
&\qquad\qquad \operatorname{Im}(z^{1/2}) > 0,\; x, x' \in (0,\omega).
\end{aligned}$$

Introducing again the factorization (4.12) of $V = uv$, one verifies as in (4.13) that

$$(H_\theta - z)^{-1} = \big(H_\theta^{(0)} - z\big)^{-1} - \big(H_\theta^{(0)} - z\big)^{-1} v\Big[I + \overline{u\big(H_\theta^{(0)} - z\big)^{-1} v}\Big]^{-1} u\big(H_\theta^{(0)} - z\big)^{-1}, \quad z \in \mathbb{C}\backslash\{\mathrm{spec}(H_\theta) \cup \mathrm{spec}(H_\theta^{(0)})\}. \tag{4.115}$$

To establish the connection with the notation used in Sections 2 and 3, we introduce the operator $K_\theta(z)$ in $L^2((0,\omega);dx)$ (cf. (2.3), (4.14)) by

$$K_\theta(z) = -\overline{u\big(H_\theta^{(0)} - z\big)^{-1} v}, \quad z \in \mathbb{C}\backslash\mathrm{spec}\big(H_\theta^{(0)}\big) \tag{4.116}$$

with integral kernel

$$K_\theta(z,x,x') = -u(x)G_\theta^{(0)}(z,x,x')v(x'), \quad z \in \mathbb{C}\backslash\mathrm{spec}\big(H_\theta^{(0)}\big),\ x,x' \in [0,\omega], \tag{4.117}$$

and the Volterra operators $H_0(z)$, $H_\omega(z)$ (cf. (2.4), (2.5)) with integral kernel

$$H(z,x,x') = u(x)g^{(0)}(z,x,x')v(x'). \tag{4.118}$$

Moreover, we introduce for a.e. $x \in (0,\omega)$,

$$\begin{aligned} f_1(z,x) &= f_2(z,x) = f(z,x) = -u(x)(e^{iz^{1/2}x} \;\; e^{-iz^{1/2}x}), \\ g_1(z,x) &= \frac{i}{2z^{1/2}} v(x) \begin{pmatrix} \frac{\exp(i\theta)\exp(-iz^{1/2}\omega)\exp(-iz^{1/2}x)}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{\exp(iz^{1/2}x)}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix}, \\ g_2(z,x) &= \frac{i}{2z^{1/2}} v(x) \begin{pmatrix} \frac{\exp(-iz^{1/2}x)}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{\exp(-i\theta)\exp(-iz^{1/2}\omega)\exp(iz^{1/2}x)}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix}. \end{aligned} \tag{4.119}$$

Introducing $\hat f_j(z,x)$, $j=1,2$, by

$$\hat f_1(z,x) = f(z,x) - \int_x^\omega dx'\, H(z,x,x')\hat f_1(z,x'), \tag{4.120}$$

$$\hat f_2(z,x) = f(z,x) + \int_0^x dx'\, H(z,x,x')\hat f_2(z,x'), \tag{4.121}$$

$$\mathrm{Im}(z^{1/2}) \geq 0,\ z \neq 0,\ x \geq 0,$$

yields solutions $\hat f_j(z,\cdot) \in L^2((0,\omega);dx)$, $j=1,2$. By comparison with (4.4), (4.5), one then identifies

$$\hat f_1(z,x) = -u(x)(\psi_+(z,x) \;\; \psi_-(z,x)), \tag{4.122}$$

$$\hat f_2(z,x) = -u(x)(\varphi_+(z,x) \;\; \varphi_-(z,x)). \tag{4.123}$$

Next we mention the following result.

Theorem 4.9. *Suppose $V \in L^1((0,\omega);dx)$, let $\theta \in [0,2\pi)$, and $z \in \mathbb{C}\backslash\mathrm{spec}\big(H_\theta^{(0)}\big)$. Then*

$$K_\theta(z) \in \mathcal{B}_1(L^2((0,\omega);dx)) \tag{4.124}$$

and

$$\det(I - K_\theta(z)) = \frac{\Delta(z) - \cos(\theta)}{\cos(z^{1/2}\omega) - \cos(\theta)}. \tag{4.125}$$

Proof. Since the integral kernel of $K_\theta(z)$ is square integrable over $(0,\omega)\times(0,\omega)$ one has of course $K_\theta(z)\in\mathcal{B}_2(L^2((0,\omega);dx))$. To prove its trace class property one imbeds $(0,\omega)$ into $\mathbb{R}$ in analogy to the half-line case discussed in the proof of Theorem 4.2, introducing

$$L^2(\mathbb{R};dx)=L^2((0,\omega);dx)\oplus L^2(\mathbb{R}\backslash[0,\omega];dx) \tag{4.126}$$

and

$$\begin{aligned}
&\tilde u(x)=\begin{cases}u(x), & x\in(0,\omega),\\ 0, & x\notin(0,\omega),\end{cases} \qquad \tilde v(x)=\begin{cases}v(x), & x\in(0,\omega),\\ 0, & x\notin(0,\omega),\end{cases}\\
&\widetilde V(x)=\begin{cases}V(x), & x\in(0,\omega),\\ 0, & x\notin(0,\omega).\end{cases}
\end{aligned} \tag{4.127}$$

At this point one can follow the proof of Theorem 4.2 line by line using (4.114) instead of (4.29) and noticing that the second and third term on the right-hand side of (4.114) generate rank one terms upon multiplying them by $\tilde u(x)$ from the left and $\tilde v(x')$ from the right.

By (4.109) and (4.111), and since

$$\det(I-K_\theta(z))=\det\Big(\big(H_\theta^{(0)}-z\big)^{-1/2}(H_\theta-z)\big(H_\theta^{(0)}-z\big)^{-1/2}\Big), \tag{4.128}$$

$\det(I-K_\theta(z))$ and $[\Delta(z)-\cos(\theta)]/[\cos(z^{1/2}\omega)-\cos(\theta)]$ have the same set of zeros and poles. Moreover, since either expression satisfies the asymptotics $1+o(1)$ as $z\downarrow-\infty$, one obtains (4.125). $\square$

An application of Lemma 2.6 and Theorem 3.2 then yields the following result relating the Fredholm determinant of $I-K_\theta(z)$ and the Floquet discriminant $\Delta(z)$.

Theorem 4.10. *Suppose $V\in L^1((0,\omega);dx)$, let $\theta\in[0,2\pi)$, and $z\in\mathbb{C}\backslash\operatorname{spec}\big(H_\theta^{(0)}\big)$. Then*

$$\begin{aligned}
\det(I-K_\theta(z))&=\frac{\Delta(z)-\cos(\theta)}{\cos(z^{1/2}\omega)-\cos(\theta)}\\
&=\bigg[1+\frac{i}{2z^{1/2}}\frac{e^{i\theta}e^{-iz^{1/2}\omega}}{e^{i\theta}e^{-iz^{1/2}\omega}-1}\int_0^\omega dx\, e^{-iz^{1/2}x}V(x)\psi_+(z,x)\bigg]\\
&\quad\times\bigg[1+\frac{i}{2z^{1/2}}\frac{1}{e^{-i\theta}e^{-iz^{1/2}\omega}-1}\int_0^\omega dx\, e^{iz^{1/2}x}V(x)\psi_-(z,x)\bigg]\\
&\quad+\frac{1}{4z}\frac{e^{i\theta}e^{-iz^{1/2}\omega}}{\big[e^{i\theta}e^{-iz^{1/2}\omega}-1\big]\big[e^{-i\theta}e^{-iz^{1/2}\omega}-1\big]}\int_0^\omega dx\, e^{iz^{1/2}x}V(x)\psi_+(z,x)\\
&\qquad\qquad\times\int_0^\omega dx\, e^{-iz^{1/2}x}V(x)\psi_-(z,x)
\end{aligned} \tag{4.129}$$

$$= \left[1 + \frac{i}{2z^{1/2}} \frac{1}{e^{i\theta}e^{-iz^{1/2}\omega} - 1} \int_0^\omega dx\, e^{-iz^{1/2}x} V(x)\varphi_+(z,x)\right]$$
$$\times \left[1 + \frac{i}{2z^{1/2}} \frac{e^{-i\theta}e^{-iz^{1/2}\omega}}{e^{-i\theta}e^{-iz^{1/2}\omega} - 1} \int_0^\omega dx\, e^{iz^{1/2}x} V(x)\varphi_-(z,x)\right]$$
$$+ \frac{1}{4z} \frac{e^{-i\theta}e^{-iz^{1/2}\omega}}{\left[e^{i\theta}e^{-iz^{1/2}\omega} - 1\right]\left[e^{-i\theta}e^{-iz^{1/2}\omega} - 1\right]} \int_0^\omega dx\, e^{iz^{1/2}x} V(x)\varphi_+(z,x)$$
$$\times \int_0^\omega dx\, e^{-iz^{1/2}x} V(x)\varphi_-(z,x). \quad (4.130)$$

Proof. Again Lemma 2.6 applies and one infers from (2.38) and (4.119)–(4.123) that

$$U(z,x) = \begin{pmatrix} 1 - \int_x^\omega dx'\, g_1(z,x')\hat{f}(z,x') & \int_0^x dx'\, g_1(z,x')\hat{f}(z,x') \\ \int_x^\omega dx'\, g_2(z,x')\hat{f}(z,x') & 1 - \int_0^x dx'\, g_2(z,x')\hat{f}(z,x') \end{pmatrix}, \quad x \in [0,\omega], \quad (4.131)$$

becomes

$$U_{1,1}(z,x) = I_2 + \frac{i}{2z^{1/2}} \int_x^\omega dx' \begin{pmatrix} \frac{\exp(i\theta)\exp(-iz^{1/2}\omega)\exp(-iz^{1/2}x')}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{e^{iz^{1/2}x'}}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix} V(x')$$
$$\times (\psi_+(z,x') \ \ \psi_-(z,x')), \quad (4.132)$$

$$U_{1,2}(z,x) = -\frac{i}{2z^{1/2}} \int_0^x dx' \begin{pmatrix} \frac{\exp(i\theta)\exp(-iz^{1/2}\omega)\exp(-iz^{1/2}x')}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{\exp(iz^{1/2}x')}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix} V(x')$$
$$\times (\varphi_+(z,x') \ \ \varphi_-(z,x')), \quad (4.133)$$

$$U_{2,1}(z,x) = -\frac{i}{2z^{1/2}} \int_x^\omega dx' \begin{pmatrix} \frac{\exp(-iz^{1/2}x')}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{\exp(-i\theta)\exp(-iz^{1/2}\omega)\exp(iz^{1/2}x')}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix} V(x')$$
$$\times (\psi_+(z,x') \ \ \psi_-(z,x')), \quad (4.134)$$

$$U_{2,2}(z,x) = I_2 + \frac{i}{2z^{1/2}} \int_0^x dx' \begin{pmatrix} \frac{\exp(-iz^{1/2}x')}{\exp(i\theta)\exp(-iz^{1/2}\omega)-1} \\ \frac{\exp(-i\theta)\exp(-iz^{1/2}\omega)\exp(iz^{1/2}x')}{\exp(-i\theta)\exp(-iz^{1/2}\omega)-1} \end{pmatrix} V(x')$$
$$\times (\varphi_+(z,x') \ \ \varphi_-(z,x')). \quad (4.135)$$

Relations (3.9) and (3.12) of Theorem 3.2 with $m = 1$, $n_1 = n_2 = 2$, $n = 4$, then immediately yield (4.129) and (4.130). □

To the best of our knowledge, the representations (4.129) and (4.130) of $\Delta(z)$ appear to be new. They are the analogs of the well-known representations of Jost functions (4.9), (4.10) and (4.55) on the half-line and on the real line, respectively. That the Floquet discriminant $\Delta(z)$ is related to infinite determinants is well-known. However, the connection between $\Delta(z)$ and determinants of Hill-type discussed in the literature (cf., e.g., [27], [14, Ch. III, Sect. VI.2], [28, Sect. 2.3]) is of a different nature than the one in (4.125) and based on the Fourier expansion of the potential V. For different connections between Floquet theory and perturbation determinants we refer to [10].

5. Integral operators of convolution-type with rational symbols

In our final section we rederive the explicit formula for the 2-modified Fredholm determinant corresponding to integral operators of convolution-type, whose integral kernel is associated with a symbol given by a rational function, in an elementary and straghtforward manner. This determinant formula represents a truncated Wiener–Hopf analog of Day's formula for the determinant associated with finite Toeplitz matrices generated by the Laurent expansion of a rational function.

Let $\tau > 0$. We are interested in truncated Wiener–Hopf-type operators K in $L^2((0,\tau);dx)$ of the form

$$(Kf)(x) = \int_0^\tau dx'\, k(x-x')f(x'), \quad f \in L^2((0,\tau);dx), \tag{5.1}$$

where $k(\cdot)$, extended from $[-\tau,\tau]$ to $\mathbb{R}\backslash\{0\}$, is defined by

$$k(t) = \begin{cases} \sum_{\ell\in\mathcal{L}} \alpha_\ell\, e^{-\lambda_\ell t}, & t>0, \\ \sum_{m\in\mathcal{M}} \beta_m e^{\mu_m t}, & t<0 \end{cases} \tag{5.2}$$

and

$$\begin{aligned} &\alpha_\ell \in \mathbb{C},\ \ell\in\mathcal{L}=\{1,\dots,L\},\ L\in\mathbb{N}, \\ &\beta_m \in \mathbb{C},\ m\in\mathcal{M}=\{1,\dots,M\},\ M\in\mathbb{N}, \\ &\lambda_\ell\in\mathbb{C},\ \mathrm{Re}(\lambda_\ell)>0,\ \ell\in\mathcal{L}, \\ &\mu_m\in\mathbb{C},\ \mathrm{Re}(\mu_m)>0,\ m\in\mathcal{M}. \end{aligned} \tag{5.3}$$

In terms of semi-separable integral kernels, k can be rewritten as,

$$k(x-x') = K(x,x') = \begin{cases} f_1(x)g_1(x'), & 0<x'<x<\tau, \\ f_2(x)g_2(x'), & 0<x<x'<\tau, \end{cases} \tag{5.4}$$

where

$$\begin{aligned} f_1(x) &= \big(\alpha_1 e^{-\lambda_1 x},\dots,\alpha_L e^{-\lambda_L x}\big), \\ f_2(x) &= \big(\beta_1 e^{\mu_1 x},\dots,\beta_M e^{\mu_M x}\big), \\ g_1(x) &= \big(e^{\lambda_1 x},\dots,e^{\lambda_L x}\big)^\top, \\ g_2(x) &= \big(e^{-\mu_1 x},\dots,e^{-\mu_M x}\big)^\top. \end{aligned} \tag{5.5}$$

Since $K(\cdot,\cdot) \in L^2((0,\tau)\times(0,\tau);dx\,dx')$, the operator K in (5.1) belongs to the Hilbert–Schmidt class,

$$K \in \mathcal{B}_2(L^2((0,\tau);dx)). \tag{5.6}$$

Associated with K we also introduce the Volterra operators H_0, H_τ (cf. (2.4), (2.5)) in $L^2((0,\tau);dx)$ with integral kernel

$$h(x-x') = H(x,x') = f_1(x)g_1(x') - f_2(x)g_2(x'), \tag{5.7}$$

such that

$$h(t) = \sum_{\ell\in\mathcal{L}} \alpha_\ell\, e^{-\lambda_\ell t} - \sum_{m\in\mathcal{M}} \beta_m e^{\mu_m t}. \tag{5.8}$$

In addition, we introduce the Volterra integral equation

$$\hat f_2(x) = f_2(x) + \int_0^x dx'\, h(x-x')\hat f_2(x'), \quad x\in(0,\tau) \tag{5.9}$$

with solution $\hat f_2 \in L^2((0,\tau);dx)$.

Next, we introduce the Laplace transform $\mathbb{F}$ of a function f by

$$\mathbb{F}(\zeta) = \int_0^\infty dt\, e^{-\zeta t} f(t), \tag{5.10}$$

where either $f \in L^r((0,\infty); dt)$, $r \in \{1,2\}$ and $\mathrm{Re}(\zeta) > 0$, or, f satisfies an exponential bound of the type $|f(t)| \leq C \exp(Dt)$ for some $C > 0$, $D \geq 0$ and then $\mathrm{Re}(\zeta) > D$. Moreover, whenever possible, we subsequently meromorphically continue $\mathbb{F}$ into the half-plane $\mathrm{Re}(\zeta) < 0$ and $\mathrm{Re}(\zeta) < D$, respectively, and for simplicity denote the result again by $\mathbb{F}$.

Taking the Laplace transform of equation (5.9), one obtains

$$\widehat{\mathbb{F}}_2(\zeta) = \mathbb{F}_2(\zeta) + \mathbb{H}(\zeta)\widehat{\mathbb{F}}_2(\zeta), \tag{5.11}$$

where

$$\mathbb{F}_2(\zeta) = \big(\beta_1(\zeta-\mu_1)^{-1}, \dots, \beta_M(\zeta-\mu_M)^{-1}\big), \tag{5.12}$$

$$\mathbb{H}(\zeta) = \sum_{\ell\in\mathcal{L}} \alpha_\ell(\zeta+\lambda_\ell)^{-1} - \sum_{m\in\mathcal{M}} \beta_m(\zeta-\mu_m)^{-1} \tag{5.13}$$

and hence solving (5.11), yields

$$\widehat{\mathbb{F}}_2(\zeta) = (1-\mathbb{H}(\zeta))^{-1}\big(\beta_1(\zeta-\mu_1)^{-1}, \dots, \beta_M(\zeta-\mu_M)^{-1}\big). \tag{5.14}$$

Introducing the Fourier transform $\mathcal{F}(k)$ of the kernel function k by

$$\mathcal{F}(k)(x) = \int_{\mathbb{R}} dt\, e^{ixt} k(t), \quad x \in \mathbb{R}, \tag{5.15}$$

one obtains the rational symbol

$$\mathcal{F}(k)(x) = \sum_{\ell\in\mathcal{L}} \alpha_\ell(\lambda_\ell - ix)^{-1} + \sum_{m\in\mathcal{M}} \beta_m(\mu_m + ix)^{-1}. \tag{5.16}$$

Thus,

$$1-\mathbb{H}(-ix) = 1-\mathcal{F}(k)(x) = \prod_{n\in\mathcal{N}}(-ix+i\zeta_n)\prod_{\ell\in\mathcal{L}}(-ix+\lambda_\ell)^{-1}\prod_{m\in\mathcal{M}}(-ix-\mu_m)^{-1} \tag{5.17}$$

for some

$$\zeta_n \in \mathbb{C},\ n \in \mathcal{N} = \{1,\dots,N\},\ N = L+M. \tag{5.18}$$

Consequently,

$$1-\mathbb{H}(\zeta) = \prod_{n\in\mathcal{N}}(\zeta+i\zeta_n)\prod_{\ell\in\mathcal{L}}(\zeta+\lambda_\ell)^{-1}\prod_{m\in\mathcal{M}}(\zeta-\mu_m)^{-1}, \tag{5.19}$$

$$(1-\mathbb{H}(\zeta))^{-1} = 1 + \sum_{n\in\mathcal{N}} \gamma_n(\zeta+i\zeta_n)^{-1}, \tag{5.20}$$

where

$$\gamma_n = \prod_{\substack{n'\in\mathcal{N}\\ n'\neq n}}(i\zeta_n - i\zeta_{n'})^{-1}\prod_{\ell\in\mathcal{L}}(\lambda_\ell - i\zeta_n)\prod_{m\in\mathcal{M}}(-i\zeta_n-\mu_m), \quad n \in \mathcal{N}. \tag{5.21}$$

Moreover, one computes

$$\beta_m = \prod_{\ell\in\mathcal{L}}(\mu_m+\lambda_\ell)^{-1}\prod_{\substack{m'\in\mathcal{M}\\ m'\neq m}}(\mu_m-\mu_{m'})^{-1}\prod_{n\in\mathcal{N}}(\mu_m+i\zeta_n), \quad m \in \mathcal{M}. \tag{5.22}$$

Combining (5.14) and (5.20) yields

$$\widehat{\mathbb{F}}_2(\zeta) = \Big(1 + \sum_{n\in\mathcal{N}} \gamma_n(\zeta + i\zeta_n)^{-1}\Big)\big(\beta_1(\zeta-\mu_1)^{-1},\dots,\beta_M(\zeta-\mu_M)^{-1}\big) \tag{5.23}$$

and hence

$$\begin{aligned}\hat f_2(x) = \bigg(&\beta_1\Big[e^{\mu_1 x} - \sum_{n\in\mathcal{N}} \gamma_n\big(e^{-i\zeta_n x} - e^{\mu_1 x}\big)(\mu_1 + i\zeta_n)^{-1}\Big],\dots \\ &\dots,\beta_M\Big[e^{\mu_M x} - \sum_{n\in\mathcal{N}} \gamma_n\big(e^{-i\zeta_n x} - e^{\mu_M x}\big)(\mu_M + i\zeta_n)^{-1}\Big]\bigg).\end{aligned} \tag{5.24}$$

In view of (3.31) we now introduce the $M\times M$ matrix

$$G = \big(G_{m,m'}\big)_{m,m'\in\mathcal{M}} = \int_0^\tau dx\, g_2(x)\hat f_2(x). \tag{5.25}$$

Lemma 5.1. *One computes*

$$G_{m,m'} = \delta_{m,m'} + e^{-\mu_m\tau}\beta_{m'}\sum_{n\in\mathcal{N}}\gamma_n e^{-i\zeta_n\tau}(\mu_m + i\zeta_n)^{-1}(\mu_{m'}+i\zeta_n)^{-1}, \quad m,m'\in\mathcal{M}. \tag{5.26}$$

Proof. By (5.25),

$$\begin{aligned}G_{m,m'} &= \int_0^\tau dt\, e^{-\mu_m t}\beta_{m'}\Big(e^{\mu_{m'}t} - \sum_{n\in\mathcal{N}}\gamma_n\big(e^{-i\zeta_n t} - e^{\mu_{m'}t}\big)(i\zeta_n + \mu_{m'})^{-1}\Big) \\ &= \beta_{m'}\int_0^\tau dt\, e^{-(\mu_m-\mu_{m'})t}\Big(1 + \sum_{n\in\mathcal{N}}\gamma_n(i\zeta_n+\mu_{m'})^{-1}\Big) \\ &\quad - \beta_{m'}\int_0^\tau dt\, e^{-\mu_m t}\sum_{n\in\mathcal{N}}\gamma_n e^{-i\zeta_n t}(i\zeta_n+\mu_{m'})^{-1} \\ &= -\beta_{m'}\sum_{n\in\mathcal{N}}\gamma_n(i\zeta_n+\mu_{m'})^{-1}\int_0^\tau dt\, e^{-(i\zeta_n+\mu_m)t} \\ &= \beta_{m'}\sum_{n\in\mathcal{N}}\gamma_n\big[e^{-(i\zeta_n+\mu_m)t} - 1\big](i\zeta_n+\mu_m)^{-1}(i\zeta_n+\mu_{m'})^{-1}.\end{aligned} \tag{5.27}$$

Here we used the fact that

$$1 + \sum_{n\in\mathcal{N}}\gamma_n(i\zeta_n+\mu_{m'})^{-1} = 0, \tag{5.28}$$

which follows from

$$1 + \sum_{n\in\mathcal{N}}\gamma_n(i\zeta_n+\mu_{m'})^{-1} = (1-\mathbb{H}(\mu_{m'}))^{-1} = 0, \tag{5.29}$$

using (5.19) and (5.20). Next, we claim that

$$-\beta_{m'}\sum_{n\in\mathcal{N}}\gamma_n(i\zeta_n+\mu_m)^{-1}(i\zeta_n+\mu_{m'})^{-1} = \delta_{m,m'}. \tag{5.30}$$

Indeed, if $m \neq m'$, then

$$\begin{aligned}
&\sum_{n\in\mathcal{N}} \gamma_n (i\zeta_n+\mu_m)^{-1}(i\zeta_n+\mu_{m'})^{-1} \\
&\quad = -\sum_{n\in\mathcal{N}} \gamma_n (\mu_m-\mu_{m'})^{-1}\big[(i\zeta_n+\mu_m)^{-1}-(i\zeta_n+\mu_{m'})^{-1}\big] = 0,
\end{aligned} \tag{5.31}$$

using (5.28). On the other hand, if $m = m'$, then

$$\begin{aligned}
\beta_m \sum_{n\in\mathcal{N}} \gamma_n (i\zeta_n+\mu_m)^{-2} &= -\beta_m \frac{d}{d\zeta}(1-\mathbb{H}(\zeta))^{-1}\bigg|_{\zeta=\mu_m} \\
&= \operatorname*{Res}_{\zeta=\mu_m}\big(\mathbb{H}(\zeta)\big)\frac{d}{d\zeta}(1-\mathbb{H}(\zeta))^{-1}\bigg|_{\zeta=\mu_m} \\
&= -\operatorname*{Res}_{\zeta=\mu_m}\frac{d}{d\zeta}\log\big((1-\mathbb{H}(\zeta))^{-1}\big) \\
&= -1,
\end{aligned} \tag{5.32}$$

using (5.19). This proves (5.30). Combining (5.27) and (5.30) yields (5.26). □

Given Lemma 5.1, one can decompose $I_M - G$ as

$$I_M - G = \operatorname{diag}(e^{-\mu_1\tau},\dots,e^{-\mu_M\tau})\,\Gamma\,\operatorname{diag}(\beta_1,\dots,\beta_M), \tag{5.33}$$

where $\operatorname{diag}(\cdot)$ denotes a diagonal matrix and the $M\times M$ matrix Γ is defined by

$$\Gamma = \big(\Gamma_{m,m'}\big)_{m,m'\in\mathcal{M}} = \bigg(-\sum_{n\in\mathcal{N}}\gamma_n e^{-i\zeta_n\tau}(\mu_m+i\zeta_n)^{-1}(\mu_{m'}+i\zeta_n)^{-1}\bigg)_{m,m'\in\mathcal{M}}. \tag{5.34}$$

The matrix Γ permits the factorization

$$\Gamma = A\,\operatorname{diag}(\gamma_1 e^{-i\zeta_1\tau},\dots,\gamma_N e^{-i\zeta_N\tau})\,B, \tag{5.35}$$

where A is the $M\times N$ matrix

$$A = \big(A_{m,n}\big)_{m\in\mathcal{M},n\in\mathcal{N}} = \big((\mu_m+i\zeta_n)^{-1}\big)_{m\in\mathcal{M},n\in\mathcal{N}} \tag{5.36}$$

and B is the $N\times M$ matrix

$$B = \big(B_{n,m}\big)_{n\in\mathcal{N},m\in\mathcal{M}} = \big(-(\mu_m+i\zeta_n)^{-1}\big)_{n\in\mathcal{N},m\in\mathcal{M}}. \tag{5.37}$$

Next, we denote by Ψ the set of all monotone functions $\psi\colon\{1,\dots,M\}\to\{1,\dots,N\}$ (we recall $N = L+M$) such that

$$\psi(1) < \dots < \psi(M). \tag{5.38}$$

The set Ψ is in a one-to-one correspondence with all subsets $\widetilde{\mathcal{M}}^{\perp} = \{1,\dots,N\}\backslash\widetilde{\mathcal{M}}$ of $\{1,\dots,N\}$ which consist of L elements. Here $\widetilde{\mathcal{M}}\subseteq\{1,\dots,N\}$ with cardinality of $\mathcal{M}$ equal to M, $|\widetilde{\mathcal{M}}| = M$.

Moreover, denoting by A_ψ and B^ψ the $M\times M$ matrices

$$A_\psi = \big(A_{m,\psi(m')}\big)_{m,m'\in\mathcal{M}}, \quad \psi\in\Psi, \tag{5.39}$$

$$B^\psi = \big(B_{\psi(m),m'}\big)_{m,m'\in\mathcal{M}}, \quad \psi\in\Psi, \tag{5.40}$$

one notices that

$$A_\psi^\top = -B^\psi, \quad \psi\in\Psi. \tag{5.41}$$

The matrix A^ψ is of Cauchy-type and one infers (cf. [24, p. 36]) that

$$A_\psi^{-1} = D_1^\psi A_\psi^\top D_2^\psi, \tag{5.42}$$

where D_j^ψ, $j = 1, 2$, are diagonal matrices with diagonal entries given by

$$\big(D_1^\psi\big)_{m,m} = \prod_{m' \in \mathcal{M}} (\mu_{m'} + i\zeta_{\psi(m)}) \prod_{\substack{m'' \in \mathcal{M} \\ m'' \neq m}} (-i\zeta_{\psi(m'')} + i\zeta_{\psi(m)})^{-1}, \quad m \in \mathcal{M}, \tag{5.43}$$

$$\big(D_2^\psi\big)_{m,m} = \prod_{m' \in \mathcal{M}} (\mu_m + i\zeta_{\psi(m')}) \prod_{\substack{m'' \in \mathcal{M} \\ m'' \neq m}} (\mu_m - \mu_{m''})^{-1}, \quad m \in \mathcal{M}. \tag{5.44}$$

One then obtains the following result.

Lemma 5.2. *The determinant of $I_M - G$ is of the form*

$$\begin{aligned} \det_{\mathbb{C}^M}(I_M - G) = (-1)^M \exp\bigg(-\tau \sum_{m \in \mathcal{M}} \mu_m\bigg) \bigg(\prod_{\ell \in \mathcal{L}} \beta_\ell\bigg) \sum_{\psi \in \Psi} \bigg(\prod_{\ell' \in \mathcal{L}} \gamma_{\psi(\ell')}\bigg) \\ \times \exp\bigg(-i\tau \sum_{\ell'' \in \mathcal{L}} \zeta_{\psi(\ell'')}\bigg) \big[\det_{\mathbb{C}^M}\big(D_1^\psi\big) \det_{\mathbb{C}^M}\big(D_2^\psi\big)\big]^{-1}. \end{aligned} \tag{5.45}$$

Proof. Let $\psi \in \Psi$. Then

$$\begin{aligned} \det_{\mathbb{C}^M}\big(A_\psi\big) \det_{\mathbb{C}^M}\big(B^\psi\big) &= (-1)^M \big[\det_{\mathbb{C}^M}\big(A_\psi\big)\big]^2 \\ &= (-1)^M \big[\det_{\mathbb{C}^M}\big(D_1^\psi\big) \det_{\mathbb{C}^M}\big(D_2^\psi\big)\big]^{-1}. \end{aligned} \tag{5.46}$$

An application of the Cauchy–Binet formula for determinants yields

$$\det_{\mathbb{C}^M}(\Gamma) = \sum_{\psi \in \Psi} \det_{\mathbb{C}^M}\big(A_\psi\big) \det_{\mathbb{C}^M}\big(B^\psi\big) \prod_{m \in \mathcal{M}} \gamma_{\psi(m)} e^{-i\tau\zeta_{\psi(m)}}. \tag{5.47}$$

Combining (5.33), (5.46), and (5.47) then yields (5.45). ☐

Applying Theorem 3.3 then yields the principal result of this section.

Theorem 5.3. *Let K be the Hilbert–Schmidt operator defined in* (5.1)–(5.3). *Then*

$$\det_2(I - K) = \exp\bigg(\tau k(0_-) - \tau \sum_{m \in \mathcal{M}} \mu_m\bigg) \sum_{\substack{\widetilde{\mathcal{L}} \subseteq \{1, \dots, N\} \\ |\widetilde{\mathcal{L}}| = L}} V_{\widetilde{\mathcal{L}}} \exp\big(-i\tau v_{\widetilde{\mathcal{L}}}\big) \tag{5.48}$$

$$= \exp\bigg(\tau k(0_+) - \tau \sum_{\ell \in \mathcal{L}} \lambda_\ell\bigg) \sum_{\substack{\widetilde{\mathcal{M}} \subseteq \{1, \dots, N\} \\ |\widetilde{\mathcal{M}}| = M}} W_{\widetilde{\mathcal{M}}} \exp\big(i\tau w_{\widetilde{\mathcal{M}}}\big). \tag{5.49}$$

Here $k(0_\pm)=\lim_{\varepsilon\downarrow 0}k(\pm\varepsilon)$, $|\mathcal{S}|$ *denotes the cardinality of* $\mathcal{S}\subset\mathbb{N}$, *and*

$$\begin{aligned} V_{\widetilde{\mathcal{L}}} &= \prod_{\ell\in\mathcal{L},\, m\in\widetilde{\mathcal{L}}^\perp}(\lambda_\ell - i\zeta_m)\prod_{\ell'\in\widetilde{\mathcal{L}},m'\in\mathcal{M}}(\mu_{m'}+i\zeta_{\ell'}) \\ &\quad\times\prod_{\ell''\in\mathcal{L},m''\in\mathcal{M}}(\mu_{m''}+\lambda_{\ell''})^{-1}\prod_{\ell'''\in\widetilde{\mathcal{L}},m'''\in\widetilde{\mathcal{L}}^\perp}(i\zeta_{m'''}-i\zeta_{\ell'''})^{-1}, \qquad (5.50)\\ W_{\widetilde{\mathcal{M}}} &= \prod_{\ell\in\mathcal{L},\, m\in\widetilde{\mathcal{M}}}(\lambda_\ell - i\zeta_m)\prod_{\ell'\in\widetilde{\mathcal{M}}^\perp,m'\in\mathcal{M}}(\mu_{m'}+i\zeta_{\ell'}) \\ &\quad\times\prod_{\ell''\in\mathcal{L},m''\in\mathcal{M}}(\mu_{m''}+\lambda_{\ell''})^{-1}\prod_{\ell'''\in\widetilde{\mathcal{M}}^\perp,m'''\in\widetilde{\mathcal{M}}}(i\zeta_{\ell'''}-i\zeta_{m'''})^{-1}, \qquad (5.51)\\ v_{\widetilde{\mathcal{L}}} &= \sum_{m\in\widetilde{\mathcal{L}}^\perp}\zeta_m, \qquad (5.52)\\ w_{\widetilde{\mathcal{M}}} &= \sum_{\ell\in\widetilde{\mathcal{M}}^\perp}\zeta_\ell \qquad (5.53)\end{aligned}$$

with

$$\widetilde{\mathcal{L}}^\perp=\{1,\dots,N\}\backslash\widetilde{\mathcal{L}} \text{ for } \widetilde{\mathcal{L}}\subseteq\{1,\dots,N\},\ |\widetilde{\mathcal{L}}|=L, \qquad (5.54)$$

$$\widetilde{\mathcal{M}}^\perp=\{1,\dots,N\}\backslash\widetilde{\mathcal{M}} \text{ for } \widetilde{\mathcal{M}}\subseteq\{1,\dots,N\},\ |\widetilde{\mathcal{M}}|=M. \qquad (5.55)$$

Finally, if $\mathcal{L}=\emptyset$ *or* $\mathcal{M}=\emptyset$, *then* K *is a Volterra operator and hence* $\det_2(I-K)=1$.

Proof. Combining (3.31), (5.43), (5.44), and (5.45) one obtains

$$\begin{aligned}\det_2(I-K) &= \det_{\mathbb{C}^M}(I_M-G)\exp\bigg(\int_0^\tau dx\, f_2(x)g_2(x)\bigg)\\ &= \det_{\mathbb{C}^M}(I_M-G)\exp\bigg(\tau\sum_{m\in\mathcal{M}}\beta_m\bigg)\\ &= \det_{\mathbb{C}^M}(I_M-G)\exp(\tau k(0_-)) \qquad (5.56)\\ &= \exp\bigg(\tau k(0_-)-\tau\sum_{m\in\mathcal{M}}\mu_m\bigg)\sum_{\substack{\widetilde{\mathcal{L}}\subseteq\{1,\dots,N\}\\ |\widetilde{\mathcal{L}}|=L}} V_{\widetilde{\mathcal{L}}}\exp\bigg(-i\tau\sum_{m\in\widetilde{\mathcal{L}}^\perp}\zeta_m\bigg),\end{aligned}$$

where

$$\begin{aligned} V_{\widetilde{\mathcal{L}}} &= (-1)^M\bigg(\prod_{m\in\widetilde{\mathcal{L}}^\perp}\beta_m\bigg)\bigg(\prod_{m'\in\widetilde{\mathcal{L}}^\perp}\gamma_{m'}\bigg)\prod_{m''\in\widetilde{\mathcal{L}}^\perp}\prod_{\substack{p\in\widetilde{\mathcal{L}}^\perp\\ p\neq m''}}(i\zeta_{m''}-i\zeta_p) \qquad (5.57)\\ &\times\prod_{p'\in\mathcal{M}}\prod_{\substack{p''\in\mathcal{M}\\ p''\neq p'}}(\mu_{p'}-\mu_{p''})\prod_{q\in\widetilde{\mathcal{L}}^\perp}\prod_{q'\in\mathcal{M}}(\mu_{q'}+i\zeta_q)^{-1}\prod_{r\in\mathcal{M}}\prod_{\substack{r'\in\widetilde{\mathcal{L}}^\perp\\ r'\neq r}}(\mu_r+i\zeta_{r'})^{-1}.\end{aligned}$$

Elementary manipulations, using (5.21), (5.22), then reduce (5.57) to (5.50) and hence prove (5.48). To prove (5.49) one can argue as follows. Introducing

$$\widetilde{\mathcal{F}(k)}(x)=\mathcal{F}(k)(-x), \quad x\in\mathbb{R} \qquad (5.58)$$

with associated kernel function

$$\tilde{k}(t) = k(-t), \quad t \in \mathbb{R}\backslash\{0\}, \tag{5.59}$$

equation (5.17) yields

$$1 - \widetilde{\mathcal{F}(k)}(x) = \prod_{n\in\mathcal{N}} (x + \zeta_n) \prod_{\ell\in\mathcal{L}} (x - i\lambda_\ell)^{-1} \prod_{m\in\mathcal{M}} (x + i\mu_m)^{-1}. \tag{5.60}$$

Denoting by $\widetilde{K}$ the truncated Wiener–Hopf operator in $L^2((0,\tau); dx)$ with convolution integral kernel $\tilde{k}$ (i.e., replacing k by $\tilde{k}$ in (5.1), and applying (5.48) yields

$$\det{}_2(I - \widetilde{K}) = \exp\bigg(\tau\tilde{k}(0_-) - \tau\sum_{\ell\in\mathcal{L}}\lambda_\ell\bigg) \sum_{\substack{\widetilde{\mathcal{M}}\subseteq\{1,\dots,N\} \\ |\widetilde{\mathcal{M}}|=M}} W_{\widetilde{\mathcal{M}}} \exp\big(i\tau \sum_{\ell\in\widetilde{\mathcal{M}}^\perp} \zeta_\ell\big). \tag{5.61}$$

Here $W_{\widetilde{\mathcal{M}}}$ is given by (5.51) (after interchanging the roles of λ_ℓ and μ_m and interchanging ζ_m and $-\zeta_\ell$, etc.) By (5.59), $\tilde{k}(0_-) = k(0_+)$. Since $\widetilde{K} = K^\top$, where $K^\top$ denotes the transpose integral operator of K (i.e., $K^\top$ has integral kernel $K(x',x)$ if $K(x,x')$ is the integral kernel of K), and hence

$$\det{}_2(I - \widetilde{K}) = \det{}_2(I - K^\top) = \det{}_2(I - K), \tag{5.62}$$

one arrives at (5.49).

Finally, if $\mathcal{L} = \emptyset$ then $k(0_+) = 0$ and one infers $\det_2(I - K) = 1$ by (5.49). Similarly, if $\mathcal{M} = \emptyset$, then $k(0_-) = 0$ and again $\det_2(I - K) = 1$ by (5.48). $\square$

Remark 5.4. (*i*) *Theorem 5.3 permits some extensions. For instance, it extends to the case where* $\mathrm{Re}(\lambda_\ell) \geq 0$*,* $\mathrm{Re}(\mu_m) \geq 0$*. In this case the Fourier transform of* k *should be understood in the sense of distributions. One can also handle the case where* $-i\lambda_\ell$ *and* $i\mu_m$ *are higher order poles of* $\mathcal{F}(k)$ *by using a limiting argument.*
(*ii*) *The operator* K *is a trace class operator,* $K \in \mathcal{B}_1(L^2((0,\tau); dx))$*, if and only if* k *is continuous at* $t = 0$ (*cf. equation* (2) *on p. 267 and Theorem 10.3 in* [12]).

Explicit formulas for determinants of Toeplitz operators with rational symbols are due to Day [7]. Different proofs of Day's formula can be found in [2, Theorem 6.29], [19], and [22]. Day's theorem requires that the degree of the numerator of the rational symbol be greater or equal to that of the denominator. An extension of Day's result avoiding such a restriction recently appeared in [6]. Determinants of rationally generated block operator matrices have also been studied in [38] and [39]. Explicit representations for determinants of the block-operator matrices of Toeplitz type with analytic symbol of a special form has been obtained in [20]. Textbook expositions of these results can be found in [2, Theorem 6.29] and [3, Theorem 10.45] (see also [4, Sect. 5.9]).

The explicit result (5.49), that is, an explicit representation of the 2-modified Fredholm determinant for truncated Wiener-Hopf operators on a finite interval, has first been obtained by Böttcher [1]. He succceeded in reducing the problem to that of Toeplitz operators combining a discretization approach and Day's formula. Theorem 5.3 should thus be viewed as a continuous analog of Day's formula. The method of proof presented in this paper based on (3.31) is remarkably elementary and direct. A new method for the computation of (2-modified) determinants for truncated Wiener-Hopf operators, based on the Nagy–Foias functional model, has

recently been suggested in [26] (cf. also [25]), without, however, explicitly computing the right-hand sides of (5.48), (5.49). A detailed exposition of the theory of operators of convolution type with rational symbols on a finite interval, including representations for resolvents, eigenfunctions, and (modified) Fredholm determinants (different from the explicit one in Theorem 5.3), can be found in [11, Sect. XIII.10]. Finally, extensions of the classical Szegő–Kac–Achiezer formulas to the case of matrix-valued rational symbols can be found in [16] and [17].

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